

# THE MATHEMATICS TEACHER

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## Why We Teach Mathematics\*

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THE question which is raised by the title that I have given this paper is one which must have been in the minds of all of you a great many times. At the present you cannot escape it if you would, for it is forced upon you by persons who are insistent that you answer it. And some of these persons have their own answer ready; they will tell you that we teach mathematics because it has become traditional to do so. Such persons may or may not admit that there was at one time a fairly good reason for including certain mathematical instruction as a fundamental part of our school programs, but with a very knowing air they will say that the schools must today "go social" in order to save humanity, and they will add that this socialization is being greatly hindered by the dead weight of such traditions as the mathematics one. And you certainly cannot reply that habits and customs often do not stand in our way. We all have complained about them, just as we complain of the Model T Fords which get in front of us on the roads ten years after the last one rolled off the assembly line. This whole question of tradition is a difficult one, but we must not accept uncritically what the rather shal-

low sociologists who often serve us our educational sociology have to say about it. If we are to appeal to sociologists let us turn to actual forthright students of society. In his *Man's Rough Road* Keller says this:

The ways of the past have always been challenged; and it is good that they were, else stagnation would have prevailed. When men are content to do just as their fathers did, they merely repeat the past, errors and all, because it is easier to imitate than to discriminate. Deference to Authority is not the same thing as deference to Experience. In every age, too, there has been resistance to change; and this also is well, for it has prevented reckless abandonment of hard-won advantages. Traditions are always falling out of adjustment to the ever changing conditions of society's life. They become out of date. On the other hand, they do not lose all their virtue overnight. The past may be a sodden, unelastic clod; it can be a springy take-off.

And again he says:

The evolutionist knows that what exists has had its justification and that the presumption is always in favor of long standing institutions. If they seem to be turning into misfits, he, having no faith in sweeping programs for their abolition must applaud the effort soberly to examine the situation in the light of knowledge of their nature and history, so as to adjust them at points where they are beginning to rattle or pinch."

\* Read at the first Mathematics Conference at Teachers College, Columbia University, July 9, 1936.

These statements of Professor Keller seem to me full of sound sense and his language is forceful and convincing. One needs merely survey the four large volumes of which he was joint author with Professor Sumner, and of which the volume from which I quoted was a condensation for the general reader, to be convinced of the extensive knowledge and balanced judgment that Professor Keller has with regard to man's general habits and traits, of which our teaching and our schools are a part. We should as teachers of mathematics be willing soberly to examine the situation in the light of knowledge and make earnest efforts to readjust our teaching at places where it may be rattling or pinching.

One thought occurs at once. What is one to do about the extreme "isms," fads, and other movements which with such frequency have swept over the educational world of late? Usually the advocates of such plans are well meaning enthusiasts who, however, cannot be considered as careful students of either society or education. Precisely because of their lack of knowledge and their tendency to adopt a sentimental or emotional viewpoint, it is difficult to find even a common basis for studying a problem with them; one often finds that he is in fact speaking almost a different language than the one they employ. Anything we have to say in answer or opposition to such views as are here indicated must be directed to other persons and not to the proposers or advocates of the views. Some of the attacks which have been made upon mathematics and an apparent movement away from it have caused alarm among teachers of the subject. I cannot share this thought if I take a somewhat long-time view of the situation. Unquestionably the importance of mathematics will be underrated here and there for a certain time, and this is likely the case at present. But if mathematics teachers are as devoted to the subject as they should be, if they study their problems as they should, if they profit from

honest and just criticism and join other teachers in helping explode superficial educational theories, there is no reason why mathematics should not maintain a position of preeminent esteem in our school system.

Mathematics teachers are more conversant than others with the great extent to which the subject is interwoven with modern life, and they have the obligation to make the fact known, undeterred by the charge, which is certain to be made, that they are not disinterested witnesses. Our position, however, is strengthened when we find persons not directly connected with the subject who endorse what we have to say. Let me call your attention to some recent instances. A few months ago Robert Quillen discussed in his newspaper "column" some requisites of an education. He spoke of the necessity of a person knowing some history, geography, and having a little acquaintance with the sciences. He ended with the statement: "But chiefly you must know your own tongue, German and French for Europe, Spanish for South America, Chinese or Japanese for the Orient, and mathematics for all undertakings everywhere." That is a pretty strong sentence, and some of our education friends might say, But what does Mr. Quillen know about 'trends' in contemporary life; has he made the appropriate surveys? I suggest that we might answer that Mr. Quillen is a cultivated and informed person. He also must know sufficiently about the important things that are going on in the world for newspapers to pay him for his daily contributions.

At present the *Literary Digest* carries from time to time an advertisement for *The Handbook of Applied Mathematics*, a book of over 1,000 pages. The advertisement states that:

Every man in business, every man in the mechanical trades, every man who ever uses a tool or has to make calculations or estimates in office or home, will find here a treasury of money-making, money-saving ideas.

It is an amazing time-saver for anyone concerned with engineering, architecture, electricity, mechanics, construction, or with accounting, auditing, manufacturing costs, taxes, or any other business mathematics. No practical man, no house-owner who makes an occasional repair, no one who has a home workshop can afford to be without the valuable information quickly found in this book.

These are strong statements and they remain convincing even if one allows a liberal discounting factor on the basis that the purpose of an advertisement is to sell something. Let us now see whether this book grows or shrinks in successive editions; in this way we shall be investigating a "trend" and so be following reputable educational practice. It is stated that the new edition is much enlarged, and that the portions dealing with pulleys, gears, use of the slide rule, and weights and measures are greatly lengthened. We are told that there are new sections on strength of materials, papering, glazing, radio, with new pages on machine shop practice and sheet metal work. Furthermore there is an exhaustive new treatment of business mathematics.

Now what is the general implication of an advertisement for a book such as the one described? Does it suggest that our schools are putting too much or too little emphasis upon mathematics? Do pupils leave the schools knowing more mathematics than they might profitably use, or less? It might be urged that the advertisement proves that mathematics teachers are failing to do what they should do, that after the public has paid to have boys and girls taught mathematics they still are ignorant of the subject. This interpretation cannot be summarily dismissed, for it deserves to be candidly examined. The advertisement mentioned an impressive number of applications of mathematics, and it does behoove us to see whether our instruction makes provision for some of the more prevalent applications, and whether our basic teaching is of a character that will allow a person to go into special fields easily. But even if there is

failure in current mathematics instruction, the cause for such failure is only in part to be laid upon the actual teachers of mathematics. Administrators, superintendents and principals must often bear a large part of the blame. If, as is now too often the case, the completion of one grade of study does not mean the actual fulfillment of any standard of achievement, and if students are by administrative mandate passed on to the next grade irrespective of what they know or can do, then teachers are freed of much of their responsibility. But this is a digression from our main thought. This book of a thousand pages is very good evidence that there is a great deal of mathematics that a large number of boys and girls should acquire when in the primary and secondary school.

Here is a good place to make a contrast. Probably some of you have seen the book *A Challenge to Secondary Education* by Everett and others, which appeared last year. The chapters in the book were written by the different members of a committee of the Society for Curriculum Study, Mr. Everett being the chairman of the committee. He himself contributed a chapter entitled "Modernizing the American High School." That name is very alluring, and doubtless was intended to be so, though I confess that I am inclined to suspect what is put forth under such a title. Now I have called your attention to the fact that Mr. Quillen, an informed man and an accomplished writer, says that in this modern world one needs mathematics for all undertakings everywhere. I have also called your attention to a recent book which makes a strong case of the necessity of mathematics. Does Mr. Everett agree with such analyses of the contemporary world and say that the modern high school must emphasize mathematics? I feel sure you are too familiar with educational literature to expect that. Here is what Mr. Everett says:

Most Americans will never use history, mathematics, physics, chemistry, and the like, as these subjects are now taught. For the most

part such skills and knowledge can be learned in a graduate school, or whenever individuals feel the real need for such specialization.

In this statement there is a qualifying phrase about the way in which the subjects enumerated are taught. But it need not be taken very seriously. Mr. Everett does not come out with a forthright demand for revision of instruction as he would if he recognized the importance of the subjects. In fact, he plainly says the subjects should be left to the graduate school or to such time as one feels the need of specialization, and this contention proves that it is not errors of teaching of which Mr. Everett is complaining.

What a contrast in point of view there is between Mr. Everett, a professional educator, writing for teachers, and Mr. Quillen, a cultivated person, writing for the lay public and seeking to raise it a little above the level of mediocrity upon which it is prone to rest. Mr. Everett condones and defends ignorance; Mr. Quillen combats it. One wonders if Mr. Everett was trying to make sly contrasts when he said *most Americans* will never use history, mathematics, physics, chemistry, and the like. Just what is the implication here of the word *American*? Did Mr. Everett mean, but through modesty refrained from saying, that Americans, being a superior people have no need of such and such subjects? Or did he mean, but through shame refrained from saying, that Americans are as a class too dumb and too lazy to use such subjects as history, mathematics, physics, chemistry, and the like?

Leaving unanswered the riddle as to what was really in Mr. Everett's mind when he wrote the word *American*, let us look at some other recent evidence as to how insistent the need for mathematics is even for Americans.

In the editorial in the May number of *The Mathematics Teacher* Professor Reeve called attention to some remarks made in a recent report of a committee of the American Chemical Society. The report stated:

There appears to be an almost general unanimity of opinion among university professors of chemistry, physics, biology and mathematics that the high school students who are now entering our universities and who have entered within the last ten years, are much inferior in preparation in mathematics and other fundamental and basic courses to similar students of a generation ago, and that the situation is tending, if possible, toward a worse condition.

The committee believed that weakness in preparation in a fundamental subject such as mathematics was very serious, and it sought to arouse members of the chemical society to take such action as they could in the matter. It is to be noted that this is a powerful organization with 17,000 members, many of whom come from industries and commercial life. It is a hopeful sign to have such groups aware of faults in our schools. Within the past two weeks I have had evidence of a further mathematical crusade on the part of the chemists. The Chairman of the Committee which made the report referred to above was Professor R. A. Gortner, chief of the division of Agricultural biochemistry of the University of Minnesota. A member of Professor Gortner's department, Mr. Charles F. Rogers, has amplified to a considerable extent the idea of the abject mathematical weakness of many students of chemistry. He sent me a manuscript copy of a paper which he had just prepared which he calls *Arithmetic and Emotional Difficulties in Some University Students*, which I hope will be published and so become generally available. Mr. Rogers' observations are based upon a number of years experience in teaching elementary quantitative chemical analysis to sophomores and upper classmen who do not intend to major in chemistry but who are taking quantitative analysis as a basis for courses in food preparation, dietetics, and plant and animal sciences. The title to his paper indicates the extent to which Mr. Rogers finds students lacking in knowledge of comparatively simple arithmetic.

The analysis that Mr. Rogers makes

extends far beyond specific arithmetic weakness of students, and takes up some very significant implications which I shall touch upon presently. Let me note now, however, by way of summary of what has been said so far, that we can find unbiased persons not teaching mathematics who are proclaiming with emphasis that people must know mathematics in order to cope with situations that educated persons should be able to handle. In spite of this, however, persons in education such as Mr. Everett, who pose as leaders, do not admit it, and the contest for proper recognition of mathematics must be made against their opposition.

Let us now consider an aspect of mathematics which a few years ago we hardly dared mention except in a whisper. Does mathematical instruction give anything more than specific mathematical knowledge? Does it in some way give a development which is broader than any skill it imparts? Can teachers of arithmetic, algebra, and geometry legitimately feel that they are giving their pupils something more than ability to solve problems or demonstrate theorems? I believe that they can, and I think that this is a claim which we should fearlessly and persistently make. I am not saying that we should endorse the old general faculty psychology; I am not saying that we should pretend that by teaching students to demonstrate theorems in geometry we shall be teaching them to reason well in all subjects which may confront them. I am not maintaining that there is some reasoning machine, which by one sort of exercise receives general perfection.

What I have in mind is something quite different from faculty psychology. Perhaps it can be described as the contribution of mathematics towards citizenship, though I dislike to use that much abused and overworked word. You are familiar with the claim made by certain educationists that mathematics is not a social study, and know that on that account it loses favor in some eyes. We readily admit that

we are not concerned with teaching facts of a political, economic, or sociological character, and we further admit that boys and girls should be given good instruction in such subjects. But such instruction does not insure that boys and girls will possess all the habits or traits which the schools should seek to build up. There is such a thing as the habit of courage, of fearlessly meeting difficult tasks, of not giving up when confronted with what is hard, a character of fortitude and resolution. Marshal Foch is quoted as saying, "Intellect, criticism—Pah! A donkey who has more character is more useful." This is a strong statement, especially when we recall the intellect which Foch possessed, and that he was a forthright scholar, and first attracted attention as a retiring but stimulating teacher. But he knew that in the last analysis such things as courage and calmness in the face of overwhelming problems were the supreme virtues. Too many educators in America have forgotten that something is to be acquired from the persistent study of substantial studies besides the objective usefulness of such studies. We hear much of the necessity of reducing failures, of the danger of creating inferiority complexes. But after boys and girls enter the activities of life and experience some of its rough competition no one shelters them from failures and no one is oversolicitous as to whether they form an inferiority complex.

Not long ago the editor of a large city paper told me that he had twenty applicants for two positions. He gave two written examinations, the first of which he made very difficult solely for the purpose of eliminating the faint-hearted. To prepare pupils for modern life requires more than equipping them with elementary facts about our social problems, which we hear stressed so much. Our schools should try to make stout-hearted workers out of boys and girls, who realize what achievement is, who will stand up to a first rate task until it is finished, who can see a distant aim and keep straight towards it

even though the work of the moment might not seem especially interesting. There was a time when this thought was prominent in education. It was strongly expressed by President Eliot in connection with the famous report of The Committee of Ten. He wrote: "Its (education's) fundamental purpose is to produce a mental and moral fiber which will carry weight, bear strain, and endure the hardest kind of labor." The word citizenship is not mentioned in that sentence, but schools will produce good citizens if they actually fulfill the aim that President Eliot mentioned.

I think we must admit that the school of thought which has for some time dominated American education either actually denies or neglects the thesis announced by President Eliot. And there is plenty of evidence of the harm that has resulted. During the years of the depression we have all seen splendid examples of persons manfully combating difficulties. But we all know that the necessity of extending relief and assistance on a large scale has helped in many cases to undermine character. There also has been much talk of freeing persons from obligations which they themselves had contracted, just because the obligations had become onerous. If in their school life boys and girls are allowed to withdraw from studies just because they are hard, if they can select entirely at will what they shall do, if they are passed from course to course and from year to year without regard to what they actually have accomplished, it requires no seer to predict that many of them will exhibit little stamina and moral fiber when they meet some of the grim trials of modern life.

Not long ago I read a splendid article by Professor Bagley in which he made an eloquent plea for substantial subject matter. In it he contrasted our schools with those of Canada, and said that tests given American children were too easy for Canadian children and had to be re-standardized before they were given to

them. A few days later I read in a newspaper an article which contrasted automobile accidents in Canada and in the United States, the figures being much to our disadvantage. Can there be a connection between these two phenomena? Personally I believe that there is. Our appalling accident record is in part at least an indication of an undisciplined, careless people, which makes light of law. The poor showing of our school children shows undisciplined boys and girls, who have no fit standards of achievement, who have not as a group learned that failure will be the inexorable result of indifference and waste of time.

What has all this to do with the teaching of mathematics? In my mind it means that we should take vigorous exception to the charge that the study of mathematics does not have an important social quality. We have allowed ourselves to be frightened away from the thesis that President Eliot announced, feeling perhaps that it was discredited by some results of psychology. We should not assert that mathematics should be given to boys and girls solely because it is hard. First of all we insist upon its high utility, it being a subject which is constantly becoming more closely interwoven with the activities of the present day world. In addition to this it is substantial, it does call for effort, it does make boys and girls face difficulties, even the wholesome and stimulating possibility of failure, and for that reason the study of mathematics can lead to development of character.

I should like to return to the paper by Mr. Rogers. Before, I spoke only of Mr. Rogers' contention that a knowledge of mathematics, chiefly arithmetic, though essential, is frequently lacking in the pupils with which he deals. But his paper, as the title to it implies, embraces a wider range of ideas. He enumerates in all 24 deficiencies in his students and he stresses such general weaknesses as lack of habit of concentrating, inability to analyze situations, to see distinctions, to separate

the relevant from the irrelevant, etc. He believes thoroughly in the thesis that general disciplinary results can accrue from the study of substantial subject matter and he stresses the importance of mathematics. I am going to quote at some length from Mr. Rogers:

There are those who question the relative value of much general arithmetical training in any schools and who would substitute for it some other subject that is perhaps more pleasing and at the time may have more popular appeal. Perhaps too, it is only rationalization on the part of the parents and others to question the relative value of apparently unused, admittedly difficult subject matter, when compared to the study of an immediately practical subject like driving an automobile safely, the making and repairing of machinery or of baking bread and cooking vegetables. A disciplined mind that can ascertain facts, arrange them about an idea, set up a system by which to treat them, and if necessary, handle with both accuracy and assurance numbers that belong with those facts, has the advantage over those minds that can conceive only of the *qualities* of things or situations, or substitute into a formula while hoping the formula fits the conditions.

The earliest introduction to exacting mental situations comes with arithmetic, which has high disciplinary value in the rigorous necessity of handling figures to obtain the same results each time the same situation is treated numerically. Emotional discipline can and should be begun by the teachers along with arithmetical training. Parental understanding and assistance are invaluable at such times.

In these statements Mr. Rogers goes far in endorsing the general disciplinary value of even simple arithmetic, the most elementary of all our mathematical subjects. We cannot overlook views of this sort when they come from a teacher of so reputable a science as chemistry. Students should be made conscious of the fact that they face a very grim civilization. There is much charity in the world, and we are making efforts to soften some of the effects of misfortune and lessen privations and want that come with old age. There are many things that give us pleasure and relaxation, and our tastes can be abundantly satisfied. We have motor cars, radios, and movies; and the foods and

products of all climates of the world are daily on our market places. But such things cannot be had without a price, and the very complexity of our civilization makes it exacting. The fact remains that people are going to have to face life with courage and must be masters to some extent of their emotions. Mr. Rogers closes his paper with this sentence, "Intensive experience in a few subjects, freedom from distractions in grade and high schools, and high standards of accomplishment in elementary work will do much to condition young people for the rigors of university and later professional life." I believe that Mr. Rogers is unduly restrictive when he speaks only of the "rigors of university and a later professional life"; the conditioning described would be wholesome for other persons as well.

Professor Gortner, who was the chairman of the original committee of the chemical society, has added to Mr. Rogers' paper some sentences very thoughtfully chosen. He says:

I have carefully read the manuscript of the above paper and have thought over its implications. I thoroughly agree with the analysis and the conclusions that the author has drawn. More than twenty years of teaching University classes in chemistry convinces me that the lack of elementary arithmetic concepts is largely responsible for the failure of many students in chemistry courses. In the American System of Secondary Education the parents of the pupils have it within their power to demand the employment of competent instructors in our grade and high schools and to insist that the pupils receive a thorough drilling in subject matter. Apparently the parents must utilize the authority which they possess if the present deplorable situation is to be remedied.

The last sentence of Professor Gortner is a restrained but nevertheless forceful suggestion that we must distrust the present controlling element in our schools. We often hear it stated that the American people want democratic education, and will not support a system that is discriminating. It is true that our people want wide opportunities for boys and girls, but the public generally is not primarily re-

sponsible for the direction that our education has taken; it has in fact been misled by persons in charge of the schools who have not courageously upheld sound ideas and high ideals. We have real evidence that our people will support public institutions with high standards. No school in the country enjoys a higher reputation than does West Point, and it is uncompromising in its standards; the public knows this and esteems it precisely for that reason. Most of the cadets are appointed by Congressmen and so are actually political appointees. But if the son of his best friend is sent home for failure, a Congressman would hardly dare raise his voice in criticism. Even the famous Whistler was dismissed because he failed in chemistry, and there is much humor but no resentment in his remark, "If silicon had been a gas, I would have been a major general." What is true of the military academy is also true of the naval academy. Standards as exacting and as inflexible as theirs would be too severe and also inappropriate for our high schools generally. But the esteem which the two schools enjoy shows that fundamentally our people can recognize values and will support institutions which they respect and will be loyal to them and point to them in pride. The tradition of West Point and Annapolis is not just an accident, it did not just happen. It is the natural result of the ideals and the character of the men who have administered those schools, that is, of their superintendents.

If we maintain that mathematics is a subject of inestimable utility, and that patient study of it throughout a sequence of years has high disciplinary values, does it follow that such study should be made universally compulsory up through, say, the ninth school year? Personally I do not think so, at least so far as the ninth year would mean the study of formal algebra. I shall go as far as anyone in insisting on the benefit that well taught mathematics can confer upon the boy or girl of good or even only fair ability. But I can

not be an uplifter, and I think that we must be realists and remember the limitations of material with which we have to work. We must first of all see to it that the great number of boys and girls who go to college should have an appropriate foundation of subject matter courses; of such courses mathematics is a primary one. We also cannot forget the many who do not go to college but go into vocations or business and who should have had some substantial mathematical preparation. I do not know just what percentage of the total high school population these two groups make, but they are the groups from which leadership at many levels and in a multitude of fields and activities must come. They give the real vital fiber to our national life. There are, however, at present a good many boys and girls in our high schools who do not come within these two categories. I do not believe that we know as yet just what we can do for inferior pupils in the way of appropriate mathematical instruction. We probably have our share of responsibility to do what we can, although personally I would not be inclined to fight for the somewhat dubious task, when I felt confident of the results both for the pupils themselves and for the public which come from efforts bestowed upon the better groups of pupils.

I suppose that most of us believe that boys and girls with obvious ability, as shown by records in the lower grades, should be required to take good ninth grade mathematics, except for strong reasons. We are seeing such a requirement disappear in many places and this naturally arouses us. It can legitimately arouse us if the administration of a school system is definitely hostile to mathematics, and preaches the doctrine that the subject is suitable only for a small group of specialists. But if the school administration has a sympathetic and enlightened view towards mathematics, and if guidance officials are really competent, I am not sure that mathematics instruction need

suffer if it is not rigidly required. In this connection I should like to call your attention to some interesting figures given in the recent National Survey of Secondary Education.

Monograph No. 19 of the survey is entitled *The Program of Studies*. In it we find an analysis of the studies of 1533 pupils who graduated in 1930 from the five Denver high schools, as related to the amount of mathematics they took. The mathematics classification shows the number of students that took the following amounts of mathematics, expressed in semester hours: 0 or 5, 10 or 15, 20 or 25, 30 or 35, 40 or 45. Nothing is said as to whether mathematics is required for the different curricula, but the figures show that 104 students, or about 7%, took no mathematics or only 5 hours. Consequently mathematics was not a requirement for graduation. On the other hand, 187 students, or 12%, took from 40 to 45 hours of mathematics. In all there were 1,234 students, or 80% of the graduating class, that took from 20 to 45 hours of mathematics. These figures are highly significant and show that mathematics was in a strong position even though it was not required for graduation. But perhaps still more significant are the analyses of each group of students which reveal the studies other than mathematics which the group in question took. As the amount of mathematics taken increased, the amount of English and social science remained practically constant, showing a slight tendency to decrease. On the other hand the amount of foreign language and science taken both steadily increased. The amount of non-academic subjects decreased from 41% in the case of those who took no mathematics or only five hours, to a little less than 10% in the case of those who took from 40 to 45 hours of mathematics. The general results I have described would probably be expected by teachers of mathematics, but it may be helpful to us to have such facts brought to light in a survey with the scope of the

one described. It is clear that boys and girls will study mathematics to a considerable amount, and it is clear that the more mathematics they take the less they are disposed to put time on nonacademic subjects, preferring to put it upon substantial intellectual studies, which can give them some actual pride in achievement. Thus mathematics emerges as an index of value and a sign of discrimination on the part of students. There are elements in the complete story which we do not have but which would be important to us. What degree of mastery of mathematics did the Denver students acquire? How would they appear in the light of the deficiencies which Mr. Rogers says he has found so often?

The reasons that I have given thus far for studying mathematics can be called prosaic reasons, and they seem to me to be the basic ones upon which our chief emphasis is to be placed. There are, however, teachers, especially some college teachers, who say that too much emphasis has been put upon utility and general disciplinary considerations. Some of these critics say that it is not strange that mathematics is under attack, for the point of view from which it has been taught has been quite wrong. I admit that we should do all that we can to broaden the view that boys and girls have as to what is useful and practical, these being two words which they are prone to use in a very limited sense. A theorem of geometry is useful and practical if it arouses one's curiosity and gives him an hour of pleasant thinking quite as well as if it helps one to lay out the foundation of a house. Our finest satisfactions as teachers will come to us when we succeed in planting in our pupils some realization, even though limited, of those qualities which mathematicians see in the subject. But this sort of teaching must supplement and cannot entirely supplant the more basic instruction, which must be kept quite close to earth if mathematics is made a subject of general instruction.

We should also try to keep before our pupils the true place that mathematics has had in the development of our civilization. We can point out that man is not the only social creature in the world. Bees, ants, and beavers are distinctly social. What definitely sets man apart is not the social instinct, but the possession of the reasoning faculty and the development of spoken and written language. The quality which reasoning should have is cogency and precision, and this quality is attained in mathematics. Thus, when boys and girls study mathematics and language they are devoting attention to a consideration of those characteristics which uniquely set man apart from other creatures. They are engaged in truly human studies. If we are content with only social teaching, as some educators appear to advocate at the present time, our education is not essentially different from that of bees in their hives, ants in their hills, and beavers in their colonies. We are all conscious of the grave problems of an economic, social, and political nature that engulf us. But their existence should not lead us to acquiesce to a view of education that is content with facile phrases and catchwords about the present social order and a changing society. We can truthfully say that no one has an adequate appreciation of human society as we know it in our western civilization if he has not studied mathematics. We should preach to our students the gospel that no well educated person should be content to know in a mere parrot-like way that mathematics has been fundamental in our civilization. He should have the ambition to know why it has been important and he cannot know why it has been important, still is important, and will be more important, unless he studies it in some fairly systematic way.

What I have said has been of an objective nature, having to do with the question—Why should mathematics be taught? Let us now consider very briefly the subjective implication of the title to this paper. Why are we as individuals engaged in teaching mathematics? Of course we are a part of society and so have a living to make. But we should also be teaching the subject because we really like it, because there is for us a fascination in mathematical study which is a dominating one in our lives. Teaching is an exhausting occupation and it has its humdrum side; there are tasks connected with our work that are in no way alluring, disguise them as we will, and which we must accept and perform merely in the spirit of honest workmen. But our fundamental interest in our subject should be apparent to every pupil who comes into our classes. We can be certain that even the dull pupils will quickly detect whether we are mere time servers or not. Furthermore if the administration of a school has an unfriendly and erroneous view towards mathematics, the able and enthusiastic teacher need not despair. It is still within his power to make honest and enthusiastic converts of those pupils that he does have, and such boys and girls while in school and later as citizens will have an influence in setting aside false educational ideas in favor of sound ones.

I began by quoting from an eminent sociologist some statements about tradition. I shall close by quoting another sentence which Professor Keller uses with regard to the subject of property, but which can justly be applied to mathematics, though I shall alter one word to lessen somewhat the scale of time. Of mathematics we can say, "It has the dignity of centuries (ages) of inestimable service to mankind."

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JOIN THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS!

## Mathematics on the Offense\*

By DR. E. J. MOULTON

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ON A number of occasions in the last year or two I have heard someone remark that "Mathematics is on the defensive." Whatever significance there may be to this remark, it is certainly a fact that in a truer sense "Mathematics is on the offensive." As a result of a consideration of the matter, I have decided to make some remarks this evening concerning the present vitality and the significance of mathematics.

First, let me recall a little of the history of mathematics. In doing this I hope that you may see what enormous strides mathematics has made, both from the point of view of general education, and from the point of view of its development as a living field of knowledge. It should become evident that mathematics is anything but a moribund body, —that, in recent years, in fact, it has been putting on a most vigorous offensive.

### DEVELOPMENT OF MATHEMATICAL EDUCATION

In my historical sketch I shall first recall very briefly the upward trend of public education in mathematics in America. At the beginning of the nineteenth century general knowledge of mathematics in this country was very meagre indeed. It is true that a President of the United States, Thomas Jefferson, was an enthusiastic student of the subject, and even contributed to the field of spherical trigonometry, but the general level of mathematical education is indicated by the fact that most Americans had scarcely heard of algebra and geometry. The few students who entered Yale in 1825 faced the following program in mathematics. Freshman—arithmetic and algebra; Sophomore—Euclid, plane trig-

onometry, mensuration, surveying, navigation; Junior—conic sections, spherics, natural philosophy, spherical trigonometry, astronomy; Senior—to take more mathematics, one must be permitted to substitute it for Hebrew or Syriac.

Without stopping to recall details of the subsequent development of popular education in mathematics it is interesting to point out by way of contrast that today perhaps one-half of our population have a speaking acquaintance with algebra and geometry, and that there are probably over a million boys and girls studying these subjects at the present time. In the last fifty years the frontier of general mathematical knowledge has advanced from the domain of arithmetic to that of algebra and geometry and even analytical geometry with its graphical presentation of numerical relationships. Moreover, at least the name of "Calculus" is known to millions of recent college students, and to some extent even higher mathematical developments are vaguely familiar among educated persons. Thus the theory of relativity has caught the attention of millions of people, so that it is said that the mathematician Einstein is one of the half-dozen most widely known men in the world today, and the daily press includes reports concerning non-Euclidian geometry, curved spaces, and so on. Problems in the theory of numbers have come within the purview of the public by way of recent articles in the *Atlantic Monthly* and other literary magazines. Finally, the applications of probability to insurance, statistics, and other fields has given this mathematical subject a conspicuous place in general public notice.

The facts which I have stated indicate that extraordinary development of general

\* Read before a joint meeting of The Men's Mathematics Club of Chicago and The Women's Mathematics Club of Chicago, March, 1935.

mathematical knowledge in our generation compared to that of our forefathers in America. The situation is similar on other continents. Never before in the world's history have there been so many mathematicians, never before has the general populace had an equal familiarity with mathematical notions.

#### DEVELOPMENT OF MATHEMATICS

Let us turn now to the development of mathematics as a body of knowledge, aside from the matter of its wide-spread dissemination. In the history of mankind the beginnings of mathematics antedate all of our written records. It seems practically certain that mathematics began its development before any known language originated, before there was a Latin language, or a Greek, or a Hebrew, or a Chinese, or a Sanskrit language. The earliest date in history positively identified is 4241 B.C., at which time the Egyptians had already developed mathematics enough to establish a calendar year of twelve months of thirty days each plus five feast days. Assuming that the Jewish calendar dates from the creation of the world according to Genesis, we see that mathematics antedates creation by at least six centuries!

Mathematics is very old but at the same time very youthful. Its vigorous development during the last two centuries is analogous to the growth of a boy between twelve and fifteen years of age. Its long past and vigorous present should presage a long and glorious future. I venture to predict that mathematics will endure ages hence, after our present American civilization has been wiped out, after Shakespeare and the English language have been forgotten, after the site of this building in which you sit has been buried deep by the shifting sands, or, as in ages past has been covered by an ice-sheet thousands of feet thick.

To turn to definite historical facts, I wish to trace briefly the very long and and slow development of mathematics

through the ages, and to indicate in contrast the tremendous burst of mathematical progress within our own time. We shall find that it is quite comparable with the development of speed of locomotion; for ages man plodded by foot, or at best utilized the speed of the horse; then recently, he developed the steam and gasoline engines, which resulted during the nineteenth century in increasing his speed of travel from ten to fifty or sixty miles an hour, and during the first third of the twentieth century from fifty to two hundred miles an hour.

As early as 3500 B.C. the Egyptians had invented a written language and from records now extant we discover that they had adopted at that time a decimal system of notation for writing numbers. During the next twenty centuries, a period of time as long as the entire Christian era, their civilization flourished, and they made appreciable progress in mathematics, and wrote mathematics books. One book written about 1650 B.C. has come down to us and is now called the Rhind papyrus. From it we discover that during those long centuries the Egyptians had developed the art of adding, subtracting, multiplying and dividing, they had learned to use fractions whose numerators were unity, they solved problems in progressions, and found areas and volumes of certain geometric figures, including as a most surprising case the volume of a truncated square pyramid. They did not, however, know a value of  $\pi$  better than  $256/81 = 3.16 \dots$ , they had not developed a logical system of geometry, and they had not invented algebra in its symbolic form. We might say that they had invented *arithmetic*, but their notations and methods were far inferior to those learned by our grammar school children today.

Contemporaneous with the Egyptian civilization was a civilization in Babylonia to which the Sumerians were chief contributors. From one of their books written about 1850 B.C. which has been recently

found and deciphered, we learn much about their mathematics. Their number system was sexagesimal, a little more complicated than that of the Egyptians, vestiges of which persist in our division of angles into degrees, minutes, and seconds. In the course of centuries they had discovered a number of geometrical theorems, including the famous theorem usually ascribed to Pythagoras who lived 1,500 years later, but they took the value of  $\pi$  to be three. Their arithmetical powers were extraordinary, their solution of certain problems, for example, amounting to the solving of a quadratic equation by the use of the formula, though our symbolism of algebra was to them unknown. They introduced the idea of mathematical tables, and used tables of products, reciprocals, squares, cubes, square roots, and cube roots. Considering the previous stage of mathematical development, their progress was remarkable, but the better students in our high schools possibly know more mathematics than did their wise men.

In passing I wish to remark that contrary to a general belief, the mathematics of these ancient peoples was not entirely "practical." To be sure, in their problems they often mention "loaves," "men," and "houses," but they doubtless recognized the purely intellectual aspect of much of their work. I cannot refrain from digressing for a moment to quote two problems at least 3,500 years old which have been handed down from the early Egyptians and which illustrate my contention. Translated into our language, they would read as follows: 1) "Divide 100 loaves among five men in such a way that the shares received shall be in arithmetical progression and that  $1/7$  of the sum of the largest three shares shall be equal to the sum of the smallest two. What are the shares?" 2) "In each of 7 houses are 7 cats, each cat kills 7 mice, each mouse would have eaten 7 ears of spelt, each ear of spelt would produce 7 hekats of grains; how much grain is thereby saved?" I think

that you will recognize the familiar ring of these ancient problems.

To return to the historical sketch, for about a thousand years after the date of the Rhind papyrus, there was, as far as we know, very little development of mathematics. Then came the rise of the Greek civilization, and with it the development of geometry as a logical system. This occurred in comparatively modern times, extending a millenium from about 600 B.C. to 400 A.D. As you know, Plato at about 400 B.C. formed an academy and made a knowledge of geometry an admission requirement. A book written by Euclid at about 300 B.C. is one of the world's masterpieces, and has perhaps gone through more editions, and been translated into more languages, and read by more people than almost any other book except for a few considered sacred by religious cults. I have not time to comment on other Greek contributions such as conic section, the theory of numbers, and mathematical astronomy. In the thousand years of their intellectual activity, they made important advances, but the whole could now be summarized in a very few volumes. With the handicap of a poor notation for numbers, and without the symbolism of algebra, their success was limited.

In the thousand years following the Greek period, mathematical progress shifted principally to the Hindus and Arabs. Their crowning achievements were the perfection of the numeral system, the study of Diophantine analysis and the development of trigonometry. On the whole their progress was not great, but it was decidedly significant, since so much of subsequent mathematical development has depended upon the remarkable economy of thought which resulted from their contributions.

Since about 1500 A.D. the center of mathematical progress has shifted to Western Europe where it has remained for the last four centuries. Outstanding early events were the introduction of

printing which made the preservation and dissemination of knowledge simple, and the use of symbols ( $+$ ,  $-$ ,  $\times$ ,  $\div$ ,  $=$ ,  $\sqrt{\phantom{x}}$ , etc.) which made still greater economy of thought possible. The discovery or invention of analytical geometry and calculus in the seventeenth century started a new epoch on its way.

During the last two centuries the progress in mathematics has been at an ever quickening pace. Deeper and deeper have the mathematicians penetrated, until most of their work now appears as unintelligible to their non-mathematical friends as the original of the Rhind papyrus. Such titles as differential equations, calculus of variations, elliptic functions, spherical harmonics, hermitian matrices, and contact transformations mean nothing except to the advanced mathematics student. Under these circumstances all we can do, probably, to inform our friends about our research activity is to tell them *how much* is being done.

We can tell them that about thirty years ago mathematicians decided to write an Encyclopedia of Mathematics for their own use, and that even though they presented their material in the brief way for which mathematicians are noted, their encyclopedia rivalled the latest edition of the Encyclopedia Britannica in its extent.

We can tell them that a History of the Theory of Numbers, a single branch of mathematics, written a decade ago by a man who is noted among mathematicians for his conciseness, occupied three large volumes.

We can tell them that the collected works of a single mathematical genius of the eighteenth century (Euler) have been assembled into a stupendous series of about forty large volumes, and that while this extraordinary output of research has never been surpassed, it has been rivalled by later geniuses such as Cayley, Cauchy, and Poincaré.

We can tell them that there have been journals devoted solely to mathematical research for over a century, that during

recent years the number of such journals has been rapidly increasing, and that, in addition to a large number of foreign mathematical journals, there are now seven American periodicals of that nature, two of which have made their appearance within the last five years, the latest making its bow this spring.

We can tell them that the output of mathematical research is so great that mathematicians have found it necessary to publish annually a large volume giving succinct accounts of the thousands of articles and tens of thousands of pages of mathematics produced, in order to keep track of the progress in the field. We can tell them that the pressure for publication is so great as compared to this output, large as it is, that hundreds of articles submitted for publication by mathematicians holding the doctorate are rejected for lack of space, or are returned for further condensation.

We can tell them that in recent years the number of graduate students in mathematics has increased very rapidly, so that the number of doctorates conferred annually in the field is now three times as great as it was a decade ago.

We can tell them that the number of members of the American Mathematical Society has increased six-fold since 1900, that research papers are read before this Society to the number of about four hundred annually, and that this Society now spends \$25,000 annually on its research publications.

We can tell them all of these things, but we *cannot* tell them the true nature of this enormous output of research without the expenditure of more time and effort than they are willing or able to give. Perhaps what we have said will convince them of the extent of the activity, but it can scarcely be hoped that they will understand its character or quality.

To summarize our historical sketch,—in the 5400 years of mathematical history, 2,000 years were used in a development of arithmetic, another 2,000 years in a de-

velopment of geometry, then 1,000 years in a development of algebra, and finally 400 years in the development of modern mathematics which is hundreds of times more extensive than all that had preceded it. In this final period the rate of production has constantly increased, being now greater than ever before, and in America the output of the last twenty years exceeds all that had gone before.

#### THE SIGNIFICANCE OF MATHEMATICS

Our friends are of course inclined to ask the human significance of all this intellectual effort. The answer must of course be suited to the questioner.

Probably for a philosopher the simple statement that the search for truth concerning laws of logic, number relationships, and geometrical forms would suffice.

Perhaps for others we should elaborate on the idea that mathematics in its higher reaches has aesthetic qualities comparable to music, poetry, and architecture, and that its development is one of the noblest phases of human culture. If man's greatest accomplishment lies in his ability to reason consecutively, then his greatest glory lies in mathematics.

For others still we should explain the central place that mathematics has occupied in the development of the natural sciences—of astronomy, physics, chemistry, engineering, and geology,—and of the tremendous import of these sciences to human thought and actions. It was mathematics, of course, which enabled Copernicus to shatter the egocentric world of his predecessors, producing a tremendous revolution in religion and philosophy. It was mathematics which enabled Newton to establish the orderliness of nature, which enabled Maxwell to formulate fundamental laws of electricity and magnetism, which aided Chamberlain to penetrate the remote past history of our Earth, and Gibbs to lay foundations for physical chemistry. Theoretical physics is becoming so mathematical that only the

mathematical specialist can contribute to its development.

For yet another group we should point to more recent developments in the application of mathematics to biology, physiology, pathology, psychology, anthropology, economics, commerce, insurance, government, war, and education. While some of these applications involve only elementary statistics, others call upon much more elaborate theories, such as probability, partial differential equations, and the calculus of variations. The study of heredity and eugenics has called forth an extensive mathematical treatment of sampling and of deviations from a norm. To study tumors of the brain a pathologist requires the solution of complicated problems in partial differential equations. To formulate a fundamental theory of economics, one requires the calculus of variations. A book on ballistics arising out of requirements during the world war resembles celestial mechanics in its intricate mathematical developments. The examinations for actuaries for insurance service are noted for their mathematical difficulties. Electrical engineers are working with the theory of matrices, students of commerce are held for proficiency in annuities and statistics, workers on the firing line in psychology and education lament the fact that their mathematical powers are not equal to the difficulties of their problems, and so on. The significant contacts of mathematics with other fields of learning are of vital importance, and are constantly becoming more numerous. In fact, we may contend that an ultimate goal in many fields is to find a mathematical foundation for their central theories. The man who discovers a law of gravitation for economics will be hailed as one of the world's greatest geniuses.

For the average man, we should perhaps quote Professor D. E. Smith of Teachers College, Columbia University:

And if you are skeptical as to the reach of mathematics in the world in which we live, consider this simple suggestion. Aside from the

propagation of the race, the most important thing in this world is education—which term is taken to include the training of the soul as well as the training of the hand and brain, and the training for eight hours or more of daily leisure as well as the eight hours or less of toil. Let us then take the science of education as a norm for measurement. Let us now imagine, if we can, that by some mighty cataclysm there should be wiped off of the face of the earth tonight every book on mathematics, every mathematical symbol of any kind, every written page or printed sheet upon which a trace of mathematics appears, and every machine for computing or recording numbers; and then let us do the same for every piece of printing or writing that has to do with the science of education. What would happen? "It is an ill wind that blows no good," and some good would unquestionably come to the world were this done. For one thing, all wars and rumors of wars would stop tomorrow, since shells of the right size could not be sent to the proper guns and the range finders would cease to operate. For another thing, we should doubtless have more attention given to real teaching because there would be some lessening of experiment, valuable though this may be in a fair percentage of cases, particularly in those in which the approximate measure of pupils' ability is concerned, and exceedingly valuable as we might make it if we would. The actual teaching would go along about as usual; very likely, however, with a little less friction for a time. But how about life beyond our scholastic walls? Every mill in the whole world would slow down and every large concern would close until it

could replace its accounts, its statistical material, its formulas for work, its measures, its tables, and its computing machinery. Every ship of the seven seas would be stricken with blindness and would wallow helpless, awaiting the probable starvation of its human burden. Not a rivet would be driven in a skyscraper in New York City, because the steel girders would have lost their numbers, Wall Street would close its portals; the engineering world would awaken tomorrow morning to a living death; the mines would be shut down; and trade would relapse to the condition of barter as in the days of savagery. It is a picture so ridiculous that we smile at the very impossibility. But it is a real picture, ridiculous though it seems,—a picture of the world sending forth the SOS for help, a hurry call for the return of the mathematics that many take so much pleasure in condemning.

We should all agree, I think that mathematics is a vital element in our present civilization, and that it is important for the race that this age-old field of thought should continue to be actively cultivated. The incalculable advantages which come from the *economy of thought* involved in its methods, the relative permanence of its achievements for the culture of the race, and its wide and constantly widening contacts with other fields of thought, give it a claim for a conspicuous and honored place in our educational system which no intelligent educator should deny.

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# Permanency of Retention of Learning in Secondary School Mathematics

By HARL R. DOUGLASS  
*Professor of Secondary Education*  
*University of Minnesota*

THERE have been many statements and disagreements and a few investigations relative to the proportions of high school mathematics once learned which will be remembered after the passage of various amounts of time.

Stokes (1931) found that his students in the University of Minnesota High School were able to make a score on the Reeve General Mathematics Composite Scale about three-fourths as high after three months of summer vacation as at the conclusion of the year's work. At the end of a twelve month interval, they still retained apparently about the same proportion of their original test score. His data also appears to indicate that the percentage retained is not materially affected by a difference between his plans of individualized self-instruction work of group instruction.

Wulff (1932) measured the retention of some of these pupils at the end of twelve, some at the end of thirty-six months and retaught them. Some were measured and retaught both at the end of twelve and at the end of twenty-four months. The Reeve Scale used by Stokes was employed. Reteaching was carried on until proficiency reached at the end of the first year of instruction was attained. The mean relearning times were as follows:

|                               |            |
|-------------------------------|------------|
| After twelve months           | 5.90 hours |
| After fifteen months          | 3.62 hours |
| After thirty-six months       | 2.79 hours |
| For the group retaught twice: |            |
| After twelve months           | 8.18 hours |
| After twenty-four months      | 4.21 hours |
| Difference                    | 3.97 hours |

Mason (1932) worked with 342 students at the University of Minnesota High

School and also used the Reeve Scale. Re-testings were made at 3, 12, 15, 24, 27, 36, 48, 60, 72, and 84 months after conclusion of the course. Her data indicate that a very high percentage of learning, 75 to 90%, is retained at least for a few years, with little or no opportunity for review or practice, and that the very large majority of the loss takes place in the first three months. Though the numbers of subjects in the various units of this experiment are small, the similarity of the results from various units invite confidence in the findings.

As opposed to the findings of Stokes and his students, White (1930) working with 326 pupils in four Baltimore high schools found great losses in the first three to five months after the completion of ninth grade algebra—losses approximating 33% to 59% as judged by the scores on two algebra tests. Corroborating the findings of Stokes, Mason, and Wulff, as well as earlier studies in other fields by Ebbinghaus, Radossawljewetch, Book, Bean and others, she found much smaller, sometimes negligible, losses after the first few months, as indicated by re-testing at 8, 9, and 16 months. There was no significant difference between the sexes in retention; the younger and more intelligent pupils remembered more; and those intending at the close of the year of instruction to study more mathematics remembered better. Conspicuous among the skills more quickly forgotten were those involving negative and fractional exponents with numbers as bases, irrational equations, extracting of square root, and similar equations including one quadratic. Best remembered were the skills involved in setting up equations,

multiplication and division of monomials, zero exponents, fractional equations with letters as bases, and linear equations.

Layton (1932) studied the retention of first year algebra by 39 girls and 12 boys, averaging somewhat higher than normal intelligence at Milne High School of the New York State Teachers College. After 11 months the average score on the repeated New York State Regents Examination in algebra was only 62% of that obtained earlier. The boys lost more than the girls of the same I.Q. Forgetting was greatest with respect to fractions and fractional equations, quadratic equations, and square root.

As measured by retestings on alternate forms of the Douglass Survey Algebra Tests, Worcester (1928) found a very high degree of retention (80 to 100% of original scores) of first semester algebra after six weeks and fifteen weeks for the material taught the first semester, probably due partly to reviews during the second semester. He found on the other hand, considerable forgetting of second semester algebra, the retest scores being only about 55% of the original.

Jackson (1931) found that 95 boys at Mt. Herman, (Mass.) High School (average I.Q. 107) tended to forget skills and procedures in algebra even though some opportunities to use them occurred. For example, on one typical unit the percentage of correct solutions was 86, 5 days later 75, and on final examination, 69.

It is quite difficult to generalize specifically from these studies. It is clear, how-

ever, that a large proportion of the forgetting takes place within a very few months and that materials forgotten within a few years can be relearned within a few hours of instruction. Investigators differ as to the amount of forgetting, the amounts forgotten within a few months ranging from 10 to 40 or 50%. This difference invites investigation as to whether, as in science, some types of things learned e.g. fundamental skills are not retained longer than materials of other types, and that the difference in results obtained by the different investigators is attributable to differences in types of tests used.

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# Mathematical Analysis in the Social Sciences

By A. C. ROSANDER

Research Fellow, General Education Board  
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IN THE following paragraphs we wish to give illustrations of elementary mathematical analysis applied to social data, the purpose being to show how these facts may be made more meaningful in terms of simple mathematical patterns.

These examples are offered to show the type of quantitative thinking which might be stressed on the secondary school level, thinking which is not less attainable because it is now entirely ignored or limited to upper college levels.

The problem before us is not merely one of bringing down from the college level the usual data and methods of social and economic statistics. The secondary school student demands that we use data which have meaning of first rate importance to the individual, or which are associated with pressing social and economic problems whose importance the student will readily see. With regard to mathematical techniques, we shall probably find little

we might do in mathematics if we set ourselves resolutely to the real task before us.

Of course whenever one attempts a mathematical analysis of certain data he always faces the question of the reliability of the data, regardless of the science in which he is working. It is admitted that the highly refined methods of analysis cannot make precise that which was crude in the original instance. The data which we used probably are as reliable as one can expect in the field of social and economic statistics. With these preliminaries we proceed to a discussion of three illustrations of mathematical analysis:

1. Distribution of national income in 1929
2. Distribution of ownership of stock in the steel industry in 1935
3. Mathematical choice and social planning

*The distribution of national income in 1929.* According to the study made by the

TABLE I  
*Distribution of Income According to Population—1929*

| Income class | Per cent of total population | Cumulative per cent | Per cent of total income | Cumulative per cent |
|--------------|------------------------------|---------------------|--------------------------|---------------------|
| Wealthy      | 0.6                          | 0.6                 | 19.7                     | 19.7                |
| Well-to-do   | 1.8                          | 2.4                 | 9.5                      | 29.2                |
| Comfortable  | 5.9                          | 8.3                 | 13.8                     | 43.0                |
| Moderate     | 13.7                         | 22.0                | 18.8                     | 61.8                |
| Minimum      | 35.7                         | 57.7                | 26.5                     | 88.3                |
| Subsistence  | 40.6                         | 98.3*               | 11.7                     | 100.0               |

\* 1.7 per cent not accounted for.

use for many of those now taught while many quantitative principles now ignored will receive considerable emphasis.

While this may seem to be a very difficult goal, it is probably not so hard to attain as might appear on first thought. At least we believe it can be done and present herewith some examples of what we mean. These examples illustrate possibilities and nothing more; they merely intimate what

Brookings Institution,<sup>1</sup> income in the United States for 1929 was distributed in the manner shown by the figures given in Table I. The first column shows the six classes of income recipients arranged in order from the richest to the poorest. The

<sup>1</sup> Maurice Leven, Harold G. Moulton, and Clark Warburton. *America's Capacity To Consume*. Washington, D. C.: Brookings Institution, 1934, p. 87. Per cent of total income was computed from data given on page 88.

TABLE II  
Comparison of Computed  $y$ -values and Original  $y$ -values

| $x$ -values<br>(population) | Original<br>$y$ -values<br>(income) | Computed<br>$y$ -values<br>(income) | Difference between<br>original and<br>computed $y$ -values | Corrected<br>Differences |
|-----------------------------|-------------------------------------|-------------------------------------|--|--------------------------|
| 0.6                         | 19.7                                | 11.0                                | -8.7   | -6.7                     |
| 2.4                         | 29.2                                | 21.8                                | -7.4   | -3.4                     |
| 8.3                         | 43.0                                | 39.9                                | -3.1   | -0.1                     |
| 22.0                        | 61.8                                | 62.6                                | +1.2   | +2.8                     |
| 57.7                        | 88.3                                | 90.6                                | +2.3   | +3.3                     |
| 98.3                        | 100.0                               | 99.9                                | 0.0  | 0.0                      |

per cent each group forms of the total population of United States in 1929 is shown in the second column, and the cumulative proportions beginning with the most wealthy class are given in the

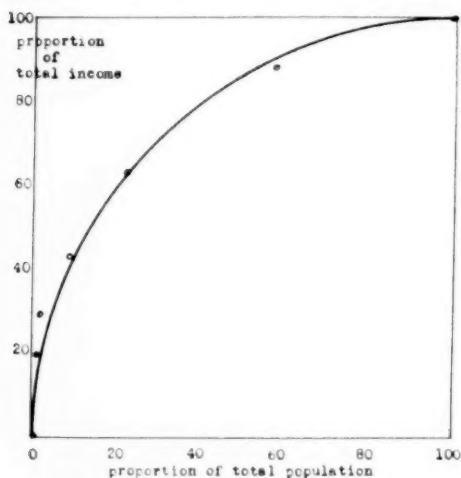


FIG. 1. Distribution of national income in United States in 1929.

third column. In the fourth column is shown the proportion of the total national income of 1929 which each class received, and in the fifth column these per cents are cumulated beginning with the richest group.

By using the cumulative per cents in Table I we can construct a double proportions curve, or a Lorenz curve as it is known in statistics. This we have done in Figure 1 where the ordinates represent cumulative proportions of total income while the abscissa represent cumulative proportions of total population.

The trend of the data immediately suggested the quadrant of a circle as an approximate fit. Since the center of this circle is at zero per cent income and 100%

population, we reversed the  $x$ -scale and obtained the following equation:

$$(x-100)^2 + y^2 = 100^2 \quad (1)$$

where  $x$  stands for the cumulative proportions of population, and  $y$  stands for the cumulative proportions of income. Simplifying and solving for  $y$  we obtain the more useful expression

$$y = \sqrt{x(200-x)} \quad (2)$$

In Figure 1 we include not only a plot of the data given in Table I but a graph of equation (2) as well. Our problem now is to determine to what extent the equation fits the data.

By substituting the  $x$ -values of Table I in equation (2) we computed a new set of  $y$ -values which are given, together with their deviations from the original  $y$ -values, in Table II. It will be seen that the equation underestimates to a rather large degree the proportions of income received in the two richest groups. The differences between the original and computed proportions of income in the other instances do not vary more than 3% one way or the other. It will be observed that while it tends to underestimate income for the rich groups it tends to overestimate that for the poorer groups. There are two possible factors to be taken into consideration in connection with the latter statement, and both deal with the data themselves.

In the first place as we have already stated in connection with Table I, there is an error in the  $x$ -values since the cumulative proportions add to 98.3% instead of to 100.0%. If this extra 1.7% belongs to the upper half of our data it will make a better fit, but if it belongs to the lower half of our data it will increase the

divergence. Of course there is no way of determining just where it does belong.

In the second place, there is a criticism of the original data by Lough in a recent book.<sup>1</sup> Lough claims that the data in Table I show too much income for the rich and too little income for the poor. Now if Lough's criticisms are true, and if the per cents of income allocated to the wealthy and to the well-to-do classes are too high in each case by 2% then we find that the deviations of the actual proportions from the computed proportions of income to be those in the last column in Table II. With the exception of the first

population increasing in multiples of 10 are given in Table III. The frequency distribution obtained from these data is shown in Figure 2.

Probably the best way to visualize Figure 2 is to imagine 10 groups of 10 individuals each, lined up in order of the amount of income received in 1929 by the entire nation from the richest group to the poorest group. If \$100 is distributed among these ten groups in the same way that it was distributed nationally in 1929, the richest group will get \$43.60 whereas the poorest group will receive only half a dollar, (\$.50); the average per member of the former group will be \$4.36 while

TABLE III  
Proportion Values Changed into Frequency Values for Frequency Distribution

| $x$<br>Cumulative<br>population<br>proportion | $y$<br>Cumulative<br>income<br>proportion<br>$y = \sqrt{x(200-x)}$ | Number of<br>dollars of<br>income |
|---|--|-----------------------------------|
| 0   | 0.0  | 00.0                              |
| 10  | 43.6   | 43.6                              |
| 20  | 60.0   | 16.4                              |
| 30  | 71.4   | 11.4                              |
| 40  | 80.0   | 8.6                               |
| 50  | 86.6   | 6.6                               |
| 60  | 92.0   | 5.4                               |
| 70  | 95.4   | 3.4                               |
| 80  | 98.0   | 2.6                               |
| 90  | 99.5   | 1.5                               |
| 100   | 100.0  | 0.5                               |

point, the deviations are not large and the average deviation is decreased from 4.5% to 3.6%. We seem justified, therefore, in stating that the equation of the quadrant  $y = \sqrt{x(200-x)}$  is a fair approximation to the distribution of national income among the total population of United States in 1929.

The next problem we faced in connection with the distribution of these data was what type of frequency distribution leads to a Lorenz curve with the equation of a quadrant. This we did by making use of equation (2) and assuming a total population of 100 and a total income of \$100. According to our equation the per cents of income which correspond to values of

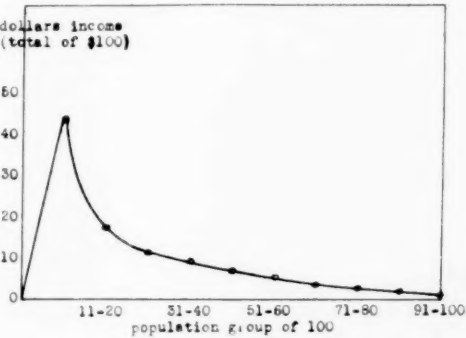


FIG. 2. Frequency distribution of income.

the average of the latter group will be a nickel, 5 cents. This means that the average income of the upper ten per cent was 87 times the average income of the lowest ten per cent. It is true that if an interval or grouping smaller than 10 is used, the distribution will be shifted, but its general shape will remain much the same, and the general results identical.

In the preceding paragraphs we have shown how the equation  $y = \sqrt{x(200-x)}$  may be used as an approximation to the data of income distribution in the United States in 1929, and how one may deduce a simple frequency distribution from this mathematical equation.

*Distribution of ownership of the steel industry.* In the issue of the *New York Times* for September 20, 1935 appeared a news item entitled "Ownership Diffused in the Steel Industry" in which was quoted, al-

<sup>1</sup> William H. Lough. *High Level Consumption*. New York: McGraw-Hill, 1935. Appendix G, p. 320 ff.

though it was not so stated, data from the September 1935 issue of *Steel Facts*, showing how stock ownership is distributed among 33 companies representing about 93% of the industry and 95% of the stockholders. The basic data are summarized below:

|  |     |     |        |              |
|--|-----|-----|--------|--------------|
| Total number of stockholders                         |     |     | 470    | 464          |
| Total number of shares                               |     |     | 39 690 | 826          |
| Number holding 1 to 20 shares                        | 301 | 002 |        | 64.0%        |
| Number holding 21 to 100 shares                      | 111 | 223 |        | 23.6%        |
| Number holding over 100 shares                       | 58  | 239 |        | 12.4%        |
| Average holding in shares                            |     |     |        | 84.4 shares  |
| Average holding in largest 100 corporations in U. S. |     |     |        | 113.9 shares |

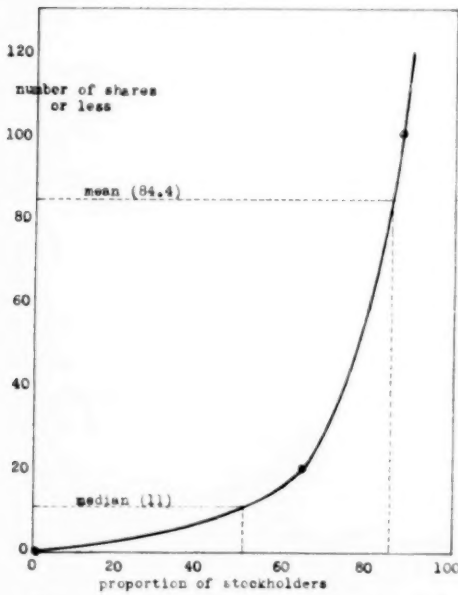


FIG. 3. Distribution of ownership in steel industry in 1935.

The *Times* headline gave the impression that ownership was diffused in the steel industry, and the headline in *Steel Facts* had a similar inference: "American Steel Industry Is Owned By Half Million Men And Women." A second bit of evidence of this impression was the statement that while the average holder of steel held "only" 84.4 shares, the average holder in the largest 100 corporations in the United States held 113.9 shares. A third piece of evidence was the picture graph of the 64%, the 24%, and the 12% groups which appeared in *Steel Facts*. Naturally the 64% group is much larger giving the impression that this is a significant group from the

viewpoint of ownership, while the smallness of the 12% group might lead the unwary to think that this group is quite insignificant. Let us analyze the data issued by *Steel Facts* and see to what extent, if any, the above quoted generalizations are justified.

While the data are not in the best form for accurate analysis, still it is possible to construct a curve showing what proportion of stockholders held less than a certain number of shares. If we use proportion of stockholders as abscissa and "less than number of shares" as ordinate we have three facts which we can use: the curve must go through the origin, the point 64% and less than 20 shares, and the point 87.6% and less than 100 shares. Such meager data make it difficult to draw an accurate curve, but we give an approximation which was found as an average between connecting these points with straight lines and a curve asymptotic to the base-line. This graph is shown in Figure 3.

From this graph it appears that approximately 85% of the stockholders own less than 84.4 shares each, the average for the group; obviously every individual of the remaining 15% own more than 84.4 shares each. The median holding, that at the 50th percentile, is approximately 11 shares according to our curve; linear interpolation between the first two points gives 16 shares.

It is possible to indicate the degree of distribution of shares by comparing the mean holding of the lower 85% group with the mean holding of the upper 15% group. We make use of the usual equation of weighted means:

$$M.N = m_1n_1 + m_2n_2$$

where  $M$  is the arithmetic mean of population  $N$ ,  $m_1$  is the mean of population

$n_1$ ,  $m_2$  is the mean of population  $n_2$ , and  $N$  is equal to the sum of  $n_1$  and  $n_2$ . What we wish to find are  $m_1$  and  $m_2$  since all the other values are given or can be computed directly. We can find  $m_1$ , the mean of the 85% group, by using the method of the center of gravity and applying it to our graph. If we do this we find the average of several approximations to be 40 shares; in other words the moments of those having between 40 and 84.4 shares are equal to the moments of those having less than 40 shares. Knowing  $m_1$  we can substitute it along with the other values in the foregoing equation and solve for  $m_2$ , the mean of the 15% group. A comparison of the number of stockholders in both the 85% group and the 15% group, and the number and proportion of shares which they own are summarized below:

85% or 399,894 stockholders own 15,995,760 shares or 40% of all shares

15% or 70,570 stockholders own 23,695,066 shares or 60% of all shares

The value of  $m_2$  is found by dividing 23,695,066 by 70,570 the quotient being 336 shares which is 8.4 times as great as the mean holding of the 85% group. Note also that the 85% group hold only 40% of the shares, not even a majority if they were pooled.

We can look at this distribution in another way. The arithmetic mean is 84.4 shares, but the median (50th percentile) is approximately 11 shares which gives some indication of the skewness of the distribution. In any symmetrical distribution the arithmetic mean and the median are equal; in our case the arithmetic mean is more than seven times as great as the median. Some idea of just what this means can be seen from the following statistical series of 11 items which has a median value of 11 and an arithmetic mean of 84: 3, 5, 7, 8, 10, 11, 40, 120, 180, 240, 300. The outstanding characteristic of such a distribution is not "diffusion" but concentration of heavily weighted values at one end.

Of course one familiar with data of this nature would not have to go through an

elaborate statistical analysis in order to conclude that the distribution of ownership of stock in the steel and iron industry was highly skewed; he could tell that by merely glancing at the data. The point we are trying to make here is that our analysis shows the outstanding and significant characteristic of this distribution to be, not diffusion as the press reports would lead us to believe, but concentration of a high degree. Incidentally as Berle and Means have pointed out in their study of the modern corporation, and as any small share holder knows, share "ownership" is a misnomer since in reality it does not represent ownership as we usually understand and use that word.

*Mathematical choice and social planning.* Our third problem deals with the mathematics of choice and its relation to social thinking and social behavior, especially with regard to social planning.

Suppose we have five red blocks numbered from one to five, and five blue blocks similarly numbered from one to five, how many possible choices are there if we pick just one block from each group? Number one red can be paired with any one of the blue blocks; that is 5 possible choices. Number two red can be paired in a similar manner; that is 5 more possible choices. So on with all the other red blocks. Thus each one of the red group leads to 5 choices and there are 5 such blocks; hence the number of possible choices is  $5 \times 5$  or  $5^2$  or 25. The chance therefore of selecting any given pair is 1 in 25.

Let us add a third group of green blocks. That means that for each of the 25 possible choices which we have just found, we would have 5 more; hence the total number of possible choices of one red, one blue, and one green block would be  $5 \times 25$  or  $5^3$  or 125. If we add a fourth group of five yellow blocks the total number of possible choices of one red, one blue, one green, and one yellow block is  $5 \times 125$  or  $5^4$  or 625. Therefore for  $n$  groups of 5 objects each the total number of possible choices is  $5^n$ .

Now we may think of our present social and economic and political organization in terms of a relatively small number of fundamental elements, each element capable of several distinct and alternative forms. Just how many mutually exclusive elements there are may be open to question; we give here a list of fifteen such elements although they overlap at certain points:

- |                   |                    |
|-------------------|--------------------|
| 1. Government     | 8. Health          |
| 2. Production     | 9. Education       |
| 3. Distribution   | 10. Religion       |
| 4. Transportation | 11. Law            |
| 5. Communication  | 12. Family         |
| 6. International  | 13. Recreation     |
| Exchange          | 14. Finance        |
| 7. Taxation       | 15. Public Welfare |

Now what we call the "social order" is the organization of all these major elements into a working whole. In most countries many of these elements have distinct lines of growth, and are often not brought into relationship one with another. In other words the so-called "social order" just grew up, like *Topsy*. On the other hand, a definite belief is growing that social organization should not be left to what is often called "evolution" which usually means let things alone, or to revolution, but should be consciously planned and integrated.

Suppose one were faced with the problem of designing a "social order," and suppose further for the sake of discussion that there were 15 main elements or building blocks, but that each of these elements had several alternative forms, only one of which could be selected. What would be the possible choices which we might make? Let us assume, and it is not an unreasonable assumption, that each of these 15 elements has 5 alternative forms from which we can choose.

This brings us back to our illustration of the possible choices in connection with the groups of 5 objects. In our present problem we must choose one of 5 distinct forms from each of 15 elements. This gives us a total number of possible choices of  $5^{15}$  or 30,517,578,125. That means that

if we make a single choice from each of these elements we have made only one choice out of a total of more than 30 billions. That would be in the same ratio as one dollar is to our present national debt. In other words the number of possible "social orders" is a magnitude of astronomical dimensions.

One could with justification argue that there are many more fundamental elements than these, that the possible forms in some cases are much greater than 5, and that each one of these forms might be analyzed into additional alternatives. In such a case our 30 billions would look very small indeed.

What are the implications of this infinity of different social patterns? One is that this method gives us a quantitative method of expressing what we usually call the "complexity of social organization." In addition we ought to realize that we have not tried many of these social patterns to date, nor should we be too confident that the peculiar one which we have is the best pattern. These enormous figures of the possible choices give a new slant on social philosophy since through a set of principles or a plan we can simplify the problem of choice. A philosophy of private ownership, private initiative, and private enterprise makes it easier for us to make selections of the peculiar forms of the fundamental elements. So does the philosophy that everything should be owned and controlled and directed by the people through their government.

*Conclusion.* Instead of the traditional mathematics on the secondary school level, we propose a type of quantitative thinking and quantitative analysis which will help every young American better to understand those social and economic problems which now face this nation. We have given three examples which to some degree at least illustrate how simple mathematical analysis might be used to make social data and social problems more meaningful.

# The Mathematics of the Sundial

By LAVERGNE WOOD and FRANCES MACK LEWIS

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MISS VEVIA BLAIR,<sup>1</sup> for many years Head of the Department of Mathematics in the Horace Mann High School for Girls, did outstanding work experimenting with new material in senior high school mathematics. She brought her unusual imagination and originality to bear on the problems of unifying the different branches of elementary mathematics, coordinating mathematics with other subjects, using the arts to make mathematics and its history vivid and satisfying, and presenting the material of elementary mathematics as a means to some immediate accomplishment. She believed that the cultural ob-

on the Sundial, she envisioned as an outlet for the knowledge gained in the study of demonstrative plane geometry, and as a means of fulfilling the objectives which she felt to be so important.

The unit was first organized and presented by Mrs. Lewis, under Miss Blair's direction, to the more able half of the eleventh grade class in the spring of 1933 and since that time has been revised and expanded for presentation to two succeeding classes. In each case the class had completed the essentials of plane geometry during the first five months of the year.

To the uninformed, the sundial is a



FIG. 1

jectives were of vital importance and planned her courses for their attainment.

The geometry of the senior high school course, Miss Blair saw and taught, not only as a subject which is its own justification but as a branch of mathematics having significant relationships with ideas and experiences in other fields. This unit

garden ornament as easily designed and set up as a bird-bath, something to be bought in an antique shop in Boston and set up in a garden in Hollywood. To the informed it is more than a garden ornament, interesting artistically and historically; it is a time-telling device, designed for a special latitude, the construction, setting, and reading of which depend

<sup>1</sup> Deceased.

on some knowledge of geography, astronomy, plane and solid geometry, and plane and spherical trigonometry.

While the unit as presented was essentially mathematical in its nature, it was necessary in order to have it culminate in the actual making of a dial, to rely on the cooperation of the teachers of fine and industrial arts. The mathematics, geography, astronomy, and history involved were taught by the teacher of mathematics. The teacher of fine arts came into the mathematics class room and gave instruction in lettering and the elements of design. The painting, etching, and cutting out of the metal were done in the art workshop under the supervision of the teacher of industrial arts. All the work was done in the hours allotted to mathematics. See Fig. 1.

#### OBJECTIVES

The making of the sundial is a centralizing purpose through which the following seem to be legitimately accomplished:

**GENERAL OBJECTIVE:** *The creative expression of mathematical and scientific knowledge through the medium of an art.*

#### MATHEMATICAL-SCIENTIFIC OBJECTIVES:

1. *Proving geometric "originals" for a purpose.*
2. *Extending concepts of two-dimensional geometry to three-dimensional geometry.*
3. *Extending concepts of two-dimensional trigonometry to three-dimensional trigonometry.*
4. *Using knowledge gained in the study of algebra, geometry, and trigonometry.*
5. *Recognizing geometry as the tool of astronomy.*
6. *Learning on a higher level, with the knowledge of the geometry of the circle and the sphere as a background, some fundamental facts of geography.*
7. *Acquiring familiarity with certain astronomical concepts.*
8. *Acquiring ability to read a sundial, making the proper corrections.*

#### ART OBJECTIVES:

1. *Acquiring skill in lettering.*
2. *Getting a knowledge of design appropriate for use on metal.*
3. *Learning the process of transferring design to metal.*
4. *Learning the process of etching.*

#### MATHEMATICAL BACKGROUND

In the year 1932-33 the sundial unit was begun in both the Tenth and Eleventh Grades, but was discontinued in the former class after a few days' trial. This class had not at that time any knowledge of geometry beyond the junior high school level. The following year, after completing the usual course in plane geometry at the end of the first five months of the school year, this same class began the sundial unit again and carried it to a successful conclusion with much enjoyment of both the mathematical and the manual work. This inability to master the fundamentals of the sundial in the earlier year may have been due in part to insufficient knowledge of geometry and in part to immaturity and lack of experience in both inductive and deductive reasoning. Successful work in this unit seems to require of the student as a foundation a knowledge of: (1) the plane geometry of the triangle, parallel lines, the circle, similar triangles, and symmetry; (2) solid geometry: concepts of the ordinary solid figures; (3) algebra: the equation, the formula, logarithms (valuable but not necessary); (4) trigonometry: the meaning of sine, cosine, and tangent, and the ability to interpolate in tables, and to make simple applications.

#### CONTENT OF THE UNIT

The two chief mathematical problems involved in the construction of the sundial are, (1) The determination of the angle at the base of the gnomon; (2) The determination of the hour angles on the face of the dial.

A general discussion of this unit, under the title, *The Sundial, a Mathematics Unit*, appears in *The Teachers College*

*Record* April 1936. It is the specific purpose of the present article to present an outline of the mathematics involved in the unit.

### I. Solid Geometry

This material was studied experimentally. Proofs were optional.

1. Intersection of two planes is a straight line.
2. A straight line perpendicular to a plane is perpendicular to every line in the plane through its foot.
3. Two lines perpendicular to the same plane are parallel.
4. Two planes perpendicular to the same line are parallel.
5. If two planes are perpendicular to each other, any line in one perpendicular to their intersection is perpendicular to the other plane.
6. If two intersecting planes are perpendicular to a third plane, their line of intersection is perpendicular to that plane.
7. If a straight line is perpendicular to a plane, every plane containing that line is perpendicular to the plane.
8. Dihedral angles
  - a. Definition
  - b. Theorems
    - (1) Two dihedral angles are equal if their plane angles are equal; and conversely, equal dihedral angles have equal plane angles.
    - (2) Any dihedral angle is measured by, or equals in degrees, its plane angle.
9. Spherical angles.
  - a. Definition
10. Spherical Triangles
  - a. Definition
  - b. Theorems
    - (1) The sum of the sides of a convex spherical triangle is less than an arc of 360 degrees.
    - (2) The sum of the angles of a convex spherical triangle is

more than 2 and less than 6 right angles.

- (3) A spherical triangle  $\nabla$  may have one, two, or three right angles or one, two, or three obtuse angles.

### II. Spherical Trigonometry

The discussion of spherical trigonometry was limited to the right spherical triangle. Here models as well as blackboard diagrams were used as they had been found helpful to the students. See Fig. 2.

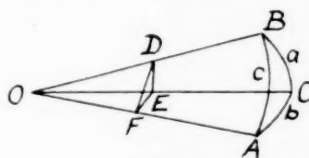


FIG. 2

1.  $O$  is the center of the sphere
2.  $ABC$  is a right spherical triangle, right angle at  $C$
3.  $OA, OB, OC$  are radii of the sphere
4. Sides  $a, b, c$ , have the same measure as the central angles  $BOC, AOC$ , and  $AOB$ , respectively.
5. Procedure
  - a. Through any point  $D$  in  $OB$ , pass a plane  $DEF$  perpendicular to  $OA$  and let it intersect the three planes  $OBC, OAC, OAB$  in the lines  $DE, EF, FD$ , respectively.
  - b. Statements
    - (1)  $DF$  and  $EF$  are  $\perp OA$   
*Reason:* (1) If a straight line is perpendicular to a plane, it is perpendicular to every line in the plane through its foot.
    - (2)  $\angle EFD = \angle A$   
*Reason:* (2) For each has same measure as the dihedral angle.
    - (3) Plane  $DEF \perp$  plane  $OAC$   
*Reason:* (3) If a straight line is perpendicular to a plane, every plane through that line is perpendicular to the plane.
    - (4) Plane  $OBC \perp$  plane  $OAC$ .  
*Reason:* (4) Angle  $C$  is a right spherical angle.

(5)  $DE \perp \text{plane } OAC$ 

*Reason:* (5) If two intersecting planes are each perpendicular to a third plane, their line of intersection is perpendicular to that plane.

(6)  $DE \perp OC$  and  $EF$ 

*Reason:* (6) If a straight line is perpendicular to a plane, it is perpendicular to every line in that plane through its foot.

- c. Thus four right angles,  $OFD$ ,  $OFE$ ,  $OED$ ,  $FED$ , and consequently four plane right triangles are obtained by means of which the formulas for the angles and sides of the spherical triangle are found.

$$\sin A = \frac{DE}{DF} \quad \sin a = \frac{DE}{OD}$$

$$\cos A = \frac{EF}{DF} \quad \cos a = \frac{OE}{OD}$$

$$\tan A = \frac{DE}{EF} \quad \tan a = \frac{DE}{OE}$$

$$\sin b = \frac{EF}{OE} \quad \sin c = \frac{DF}{OD}$$

$$\cos b = \frac{OF}{OE} \quad \cos c = \frac{OF}{OD}$$

$$\tan b = \frac{EF}{OF} \quad \tan c = \frac{DF}{OF}$$

From these the following formulas are determined.

$$\sin A = \frac{\sin a}{\sin c}; \quad \cos A = \frac{\tan b}{\tan c};$$

$$\tan A = \frac{\tan a}{\sin b}$$

$$\cos c = \cos a \times \cos b$$

- d. By analogy the following results would be obtained. The procedure begins by taking any point in  $OA$  and passing a plane through it perpendicular to  $OB$ .

$$\sin B = \frac{\sin b}{\sin c} \quad \cos B = \frac{\tan a}{\tan c}$$

$$\tan B = \frac{\tan b}{\sin a}$$

6. Compare these formulas with those for the plane right triangle.
7. The solution of the spherical triangle is used in geodetic surveying and in all situations where the curvature of the earth must be taken into consideration. It is also used in astronomy and it is for that purpose that these formulas were developed.

### III. Theory of the Horizontal Sundial

The explanation of the horizontal sundial is simplified by using the notion of the Greeks (Apollonius, Hipparchus, and Ptolemy) and of the Middle Ages (up to Copernicus, 1473-1543) that the earth is the center of the universe. Working with this hypothesis, the astronomer and the mathematician have been able to evolve a mathematical explanation.

The heavens at night appear to be an inverted bowl. This is half of what is called the celestial sphere. The plane of the earth's equator extended is the equatorial plane of the celestial sphere; the axis of the earth extended is the axis of the celestial sphere about which all the stars seem to revolve. The North Pole star, Polaris, is seemingly fixed.

The sun moves in its path, called the ecliptic, about 4 minutes per day behind the stars, whose diurnal motion is strictly uniform, so that at the end of the year it finds itself back at the same place, having made a complete revolution of the heavens relatively to the stars from west to east. The ecliptic is inclined at an angle of  $23.5^\circ$  to the equator of the celestial sphere, but for our purposes we may consider that the sun moves in the equatorial path. The error in this assumption is not larger than 16.5 minutes.

At the equator of the earth, therefore, we may assume that the sun moves  $15^\circ$  every hour, with the rays of the sun hit-

ting the earth perpendicularly. As we move away from the equator, the rays of the sun hit the surface of the earth obliquely. Because of this, the angles on a sundial are not evenly spaced. And to discover just what these angles are at any place on the earth's surface requires further study. See Fig. 3.

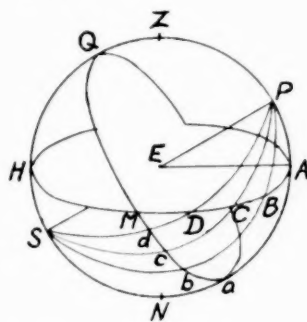


FIG. 3

In the above figure,<sup>2</sup>

1.  $P E S$  is directed toward  $N$  and  $S$  poles of heavens.
2.  $H D A$  represents the horizon plane
3. Plane  $Q d a$  is perpendicular to  $P S$ , thereby coinciding with the plane of the equator.
4.  $Z N$  represents the zenith-nadir line which is perpendicular to the horizon plane.
5.  $Q Z P$  represents the meridian of the place under consideration and by its intersection with the horizon circle will determine the 12 o'clock line  $E A$ .
6. The equatorial circle is divided into 24 equal parts. Beginning with the meridian  $Q Z P$ , through the various points of division and the poles, great circles  $P b S$ ,  $P c S$ ,  $P d S$ , etc., were drawn. The shadow of the style placed in the position of  $P E S$  will fall on these circles after successive intervals of 1, 2, 3, etc., hours from noon. If they meet the horizontal circle in points  $B$ ,  $C$ ,  $D$ ,

etc., then  $E B$ ,  $E C$ ,  $E D$ , etc., will be the hour lines required.

7. The problem of the horizontal sundial consists in calculating the angles which these lines form with the 12 o'clock line  $E A$ , whose position is known for any particular place.

a. The spherical triangles  $P A B$ ,  $P A C$ , etc., enable us to do this readily. They are all right angled at  $A$ . ( $Z N$  is perpendicular to horizon plane, so any plane containing it,  $Z P A$ , is perpendicular to horizon plane.) The side  $\widehat{P A}$  is the angle of elevation of the North Star. The angles  $A P B$ ,  $A P C$ ,  $A P D$ , etc., are respectively  $15^\circ$ ,  $30^\circ$ ,  $45^\circ$ , etc.

b. Then

$$\tan \widehat{A B} = \tan 15^\circ \sin \widehat{P A}$$

$$\tan \widehat{A C} = \tan 30^\circ \sin \widehat{P A}$$

$$\tan \widehat{A D} = \tan 45^\circ \sin \widehat{P A}$$

These formulas determine the sides  $\widehat{A B}$ ,  $\widehat{A C}$ , etc., and consequently the angles  $A E B$ ,  $A E C$ ,  $A E D$ , etc.

The calculations of these angles must extend throughout one quadrant from noon to 6 o'clock but need not be carried further, because all others can be deduced from these. The dial is symmetrically divided by the meridian and therefore hours equidistant from noon will have their hour lines equidistant from the meridian. Hour lines between six in the evening and six in the morning are prolongations of the other twelve.

8. If the imaginary sphere with all of its circles is removed and the style  $E P$  and the horizon plane with the lines traced on it are retained, the HORIZONTAL SUNDIAL remains.

9. Before the formula for the sundial can be completed, the angle of ele-

<sup>2</sup> P. 151, figure 2, The Encyclopedia Britannica, Eleventh Edition, Vol. VIII, Cambridge, Eng.: At the University Press, 1910.

vation of the North Star,  $\angle AEP$ , must be proved equal to the latitude of the observer.

The radius of the earth is approximately 4,000 miles, which is so small in comparison with the distances with which we are working as to be negligible. (Polaris, the North Star, is 466 light years away. One light year is the product of the speed of light and the number of seconds in a year—186,000  $\times$  31,560,000—.) Therefore, a line from the earth's surface to the North Star may be considered parallel to the axis of the earth.

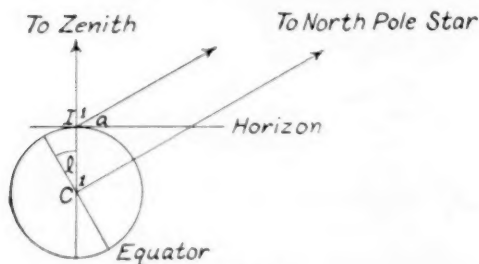


FIG. 4

- a. In the above figure (Fig. 4) (1) the observer is placed on top of the earth at  $I$ . (2)  $a$  is the angle of elevation of the North Star (3)  $l$  is the latitude of the place where the observer is located. (4)  $C$  is the center of the earth.
- b. Then

$$(1) \angle I_1 = \angle C_1$$

*Reason* (1) Corresponding angle of parallel lines are equal.

$$(2) \angle I_1 + \angle a = 90^\circ$$

*Reason* (2) Zenith line is perpendicular to the horizon.

$$(3) \angle C_1 + \angle l = 90^\circ$$

*Reason* (3) Axis of the earth is perpendicular to the equator.

$$(4) \angle a = \angle l$$

*Reason* (4) Complements of equal angles are equal.

10. The formula for the horizontal sundial can now be written  $\tan b_n^\circ = \tan 15^\circ \times \sin l$ , where  $n$  represents the hour and  $l$  the latitude of the observer.

#### 11. Vertical South Dial

A pupil interested in making a vertical south dial may proceed in the same way to derive the formula, that is, by drawing the diagram for the celestial sphere with the equatorial plane and the zenith-nadir plane facing due south, on which the angles are found. The formula for the vertical south dial is,  $\tan b_n^\circ = \tan 15^\circ \times \cos l$ .

#### IV. Construction of the Horizontal Sundial

##### 1. The Gnomon

Before the construction of the dial is begun, the decision as to the place where the dial is to be set must be made. This is because the angle at the base of the gnomon must equal the terrestrial latitude of the place in which the dial is to be used. The gnomon may be easily designed in the following way so as to have the correct angle. Draw the line  $AC$  of any desired length, according to the size of the dial to be constructed. At  $C$  construct a line perpendicular to  $AC$ . Mark off on this line  $CB$  equal to  $AC \times \tan l$  and draw  $AB$ . Angle  $BAC$  will be the required angle. See Fig. 5.

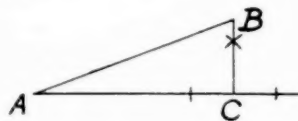


FIG. 5

##### 2. The Dial

The size of the dial must be decided upon first (an eight- or nine-inch square is very satisfactory) and next the position of the gnomon on the dial. The gnomon may be placed at the center of the dial, or it may be

located nearer the circumference, provided the hour angles are measured from that place. The width of the gnomon must be allowed for, as in the morning it is one edge that casts the shadow, and in the afternoon, the other. The design must be made on a thin but durable paper and traced on the metal. Commercial bronze has been found satisfactory.

A device for determining the hour angles is illustrated in Fig. 6 below. The angles obtained should be checked by the formula,  $\tan b_n^\circ = \tan 15^\circ \times \sin l$ . When the hour angles and the position and size of the gnomon are fixed, the shape, the design, and the motto are left to the student. It is well to consult both the teachers of fine and industrial arts, for it is important that the design and its arrangement be artistically satisfying and appropriate, and suitable to the medium in which the work is to be done.

Substituting, we have  $\tan \angle CAI = \tan 15^\circ \cdot \sin l$ .

After the design is traced on metal, the part which is not to be etched out, is painted with a paint, such as Colorite, which the acid does not attack. Each line on the dial must have a definite width either to be painted or etched out. An appropriate acid, a solution of nitric in the case of commercial bronze, is used for the etching. The metal is then cleaned with alcohol to remove the paint, then polished with steel wool and finally with a soft polish. The gnomon may be either soldered or bolted on. The sundial is now completed and ready to be set.

#### V. Setting the Dial

From the work with the celestial sphere the students know that the gnomon of the dial must be placed in the meridian plane and the dial itself on the horizon plane. These two planes must be determined for

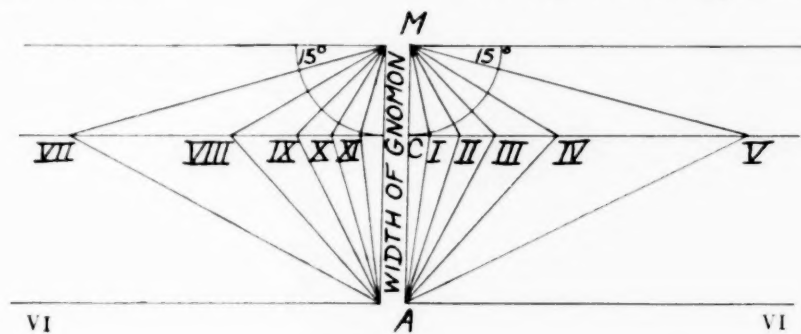


FIG. 6

In the above figure<sup>3</sup>

$MC = CA \sin l$ , by construction

In  $\triangle MCI$ ,  $\frac{IC}{MC} = \tan 15^\circ$

$$IC = MC \cdot \tan 15^\circ$$

$$IC = CA \cdot \sin l \cdot \tan 15^\circ$$

In  $\triangle CAI$ ,  $\frac{IC}{AC} = \tan \angle CAI$

$$\frac{IC}{CA} = \tan 15^\circ \cdot \sin l$$

the place where the dial is to be set.

The horizon plane is easily determined by using a carpenter's level.

The meridian plane is a little more difficult to determine. There are four ways to do this.

1. Set at noon with a good watch whose error on solar time is known. One must remember, however, that our watches and clocks tell MEAN time; the sundial tells APPARENT time. Mean time - Apparent time = equa-

<sup>3</sup> Adapted from Jacoby's *Astronomy* by permission of The Macmillan Company.

tion of time. One must also remember that in the United States we have Standard time and in the Summer months, in some sections, Day-light Saving time.

2. Compass reading. If the variation of the compass is known, i.e. the difference between the true and the magnetic north.
3. Star observation. The Gnomon should point to the North Star. The meridian goes through the North Star and the zenith.

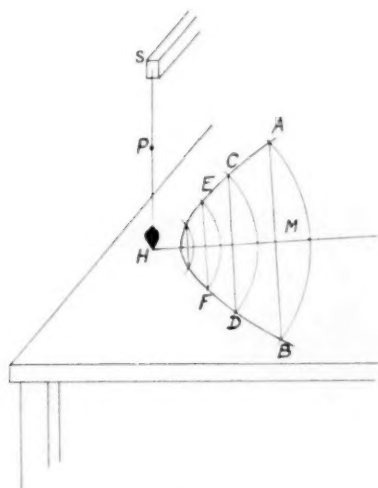


FIG. 7

4. By experiment.<sup>4</sup> See Fig. 7.
  - a. The stand on which the dial is to set must be in the sun all day. It must be level and secure, so that there is no danger of shifting.
  - b. Suspend plumb bob from point S which must be rigidly fixed.
  - c. H, where it meets the surface, should be somewhere near the middle of one end. With it as center, describe any number of concentric arcs of circles, as AB, CD, EF, etc.
  - d. A bead P, kept in its place by friction, is threaded on the plumb

bob line at some convenient height above H.

- e. Follow the shadow of P as it moves across the surface during the day. It will be found to describe a curve ACE . . . FDB (an hyperbola). At the moment when it crosses the arc AB, mark Point A. AP is then the direction of the sun and AH is the horizontal. Angle PAH is then the altitude of the sun. In the afternoon mark B when it crosses the same arc. Then angle PBH is the altitude of the sun. But the right angled triangles PHA and PHB are equal and it follows that the two positions will be symmetrically placed on either side of the meridian. Therefore, if we draw AB and find its perpendicular bisector, HM, we determine the meridian plane.
- f. Other readings should be taken to check the work. The mean of the positions is used.
- g. The most favorable time to do this is at the end of June and December when the sun's declination is almost stationary.

#### OPPORTUNITY FOR CREATIVENESS AND INDIVIDUALITY IN THIS PROJECT

The mathematics of this unit has been required of all pupils who have taken the course. The actual construction of dials has been optional, but four-fifths of the pupils have designed and constructed them. The others have substituted source-themes on such related topics as, "Time and its Measurement," "The History of the Sundial," "Time-telling Devices," or have made solid models for use of the class. Three girls elected to substitute several weeks of work in conventional demonstrative solid geometry. Each girl who elected to make a dial selected the shape of the background on which she wished to set up her design, circular, square, hexagonal, or octagonal. (See Fig. 1.) This was cut from a piece of commercial bronze.

<sup>4</sup> P. 152, fig. 6, Encyclopedia Britannica, Eleventh Edition, Vol. VIII, Cambridge, Eng.: At the University Press, 1910.

She drew a design which seemed to her beautiful and appropriate, either pictorial or conventional, drew in the hour angles which she had been able to determine by the use of the mathematics learned in this course. With this also appeared a motto of her choice, an appropriate quotation, in many cases from one of the foreign languages which she had studied. The completed design she then transferred to the metal and etched on its surface. So her dial became very completely her own creation.

#### SATISFACTIONS OF THE UNIT

This unit was satisfying from the standpoint of the teacher and the pupil because it gave opportunity for:

- (1) The unifying of the several branches of elementary mathematics and the extension of the concepts of plane geometry and trigonometry to the three-dimensional field.
- (2) The coordinating of work in several fields: mathematics, geography, astronomy, chemistry (of etching), history (of the measurement of time), and in fine and applied art.
- (3) The experience of seeing a long period of study and work culminate in a personal and concrete product.

- (4) The creation of a unique object of beauty and permanence.

An evaluation of this unit was of necessity delayed until the sundials had been set in the locations for which they had been constructed, and reports in regard to their functioning had been received. These reports over a period of three years have indicated that with three exceptions the dials are set and functioning satisfactorily in their permanent locations at various points from Los Angeles, California, to Yarmouth, Nova Scotia.

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The following issues of the *Mathematics Teacher* are still available and may be had from the office of the *Mathematics Teacher*, 525 West 120th Street, New York.

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# THE ART OF TEACHING



## A NEW DEPARTMENT

### Classroom Practice\*

By JOSEPH B. ORLEANS

1. The early development of the *proper habits and attitudes* in the pupils concerning classroom routine is essential.
  - a. Give the classroom the atmosphere of a workshop.
  - b. Train the pupils to settle down to work as soon as they enter the room without being called to order by the teacher.
  - c. Train the pupils to look for the home assignment on the same corner of the board every day.
  - d. Make use of the *class exercise* as frequently as possible, every day during the first month or six weeks, at least three times a week after that. The class exercise, given during the first five minutes of the period may serve either as a test to be rated by the teacher or by a selected pupil, or for practice. The exercise should appear on the board together with the home assignment, so that the pupils will lose no time in beginning their work. Paper need not be distributed for this. Pupils should use their own paper. The few minutes devoted to this exercise allows the teacher the opportunity to mark attendance, check home work papers, says a word to an absentee who has returned, etc.
2. Insist upon neat *written work* both on the board and on paper. Many teachers have found it helpful to have all papers folded in booklet form, with scratch work done in a margin on the right.

The pupil should understand that work written on the board is not merely for himself but belongs to the entire class. Therefore, it should be legible, so that all may be able to read it.
3. Make yourself conscious of the *physical condition of the room*, ventilation, condition of floors, bulletin board and teacher's desk. Insist that pupils cover their books.
4. Pay particular attention to your own *use of English* and to the *oral work of the pupils*. Insist upon accurate statements. Not all answers need be given in complete sentence form; but when a lesson requires it, insist upon correct English. In mathematics, more so than in other subjects, every word uttered is important and the pupils must say exactly what they mean and not leave it to the teacher to supply the deficiencies.
5. It is suggested that *papers be passed by rows* instead of by columns. This requires only a hand motion from side to side and avoids the noise and confusion caused by pupils turning around and rising from their seats when papers are passed by columns. This may seem to some teachers a trivial matter, but it is important in connection with ordinary classroom discipline.
6. *Work that has been put on the board* need not be explained necessarily by the pupil whose name it bears. To be able to explain what another person has written or to locate his error is as important as to be able to do the exercise

\* The following is a circular on Classroom Practice given to the members of the mathematics department of the George Washington High School, New York City.

oneself. This method of procedure tends to hold the attention of the pupils and to help the teacher in the "control" of the class.

7. *In geometry, theorems and exercises on the board* should ordinarily show only diagram, hypothesis and conclusion. The teacher may ask also for the statements in the proof. Reasons should never appear on the board. There is no value in having pupils merely read reasons from the board. These should be given orally. The omission of the reasons will also avoid crowded illegible handwriting.
8. The teacher must give the *home assignment* very careful consideration. Even though it has been put on the board at the beginning of the period, reference should be made to it during the period, so that the pupils will see its relation to the work on hand. The purpose of the home assignment is merely to enable the pupil to tell whether or not he has mastered the work that has been taught and to enable the teacher to tell whether or not he is to go on with the new topic. The assignment should not be too long. Home work papers should not be rated for the purpose of recording the mark.

This would tend to encourage copying. There are many methods of handling home work in the class room, depending upon the nature of the work. The teacher must use his good judgment in selecting the proper method; but he must not make a habit of having the home work examples recited on in the classroom every period.

9. *Teacher-pupil activity.* "Pupils would do better if left to wrestle more for themselves. . . . In the past we have all tended to teach too much. . . . A clever teacher who loves teaching for its own sake may be something of a danger. He may do too much of the thinking and leave the boys too little to do for themselves."—F. W. WESTAWAY, *Craftsmanship in the Teaching of Elementary Mathematics*.

A class period has been successful to the extent to which the pupils have learned the lesson; and pupils do not learn well in any grade of mathematics by merely following a demonstration by the teacher at the board. The class must be actually engaged at their desks at the same time with paper and pencil while they are thinking through the various steps in the presentation.

## PLAYS

Back numbers of *The Mathematics Teacher* containing the following plays may be had from the office of *The Mathematics Teacher*, 525 West 120th Street, New York.

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 The Eternal Triangle. Gerald Raftery.  
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The above monograph which will be the first of a series to be published soon by THE MATHEMATICS TEACHER will be sent postpaid to all subscribers whose subscriptions are paid up to November 1, 1936. The book will be sent postpaid to others at 25¢ each.

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## IN OTHER PERIODICALS



By NATHAN LAZAR

Alexander Hamilton High School, Brooklyn, New York

### Algebra and Trigonometry

1. Blake, Sue Avis. *Some of nature's curves. I: Conic sections.* School Science and Mathematics. 36: 245-49. March 1936.

A convenient summary of the occurrence of conic sections in natural phenomena.

2. Fryer, Oscar G. *The right angle slide rule.* School Science and Mathematics. 36: 422-25. April 1936.

A clear description, with the help of a diagram, of a right angle slide rule that can be used for two purposes:

- a. "All trigonometric functions can be found on it.
- b. "Any problem involving the relation of the sides of a triangle can be solved by its use."

The author also claims that "it is indispensable for solving alternating current problems where resistance, inductance and capacitance are involved," and illustrates his point by a sample calculation of a simple network.

3. Fulkerson, Elbert. *Teaching thought problems in ninth grade algebra.* School Science and Mathematics. 36: 393-98. April, 1936.

By the term "thought problem" the author refers to "those problems whose solution will depend upon equations which the pupils must derive from the facts given in the problem."

The following steps are recommended:

- a. Translating verbal statements into algebraic symbols.
- b. Eliminating irrelevant terms from the problem.
- c. Reducing the wording to enumerated statements.
- d. Translating the statements into symbolic language.
- e. Checking the results obtained.

While the writer of the article shows good insight into the difficulties faced by the beginning student of algebra, he does not seem familiar with the literature on the subject of teaching verbal problems. For nearly all the recommendations he made have already appeared in various articles in *School Science and Mathematics*, in *The Mathematics Teacher*, not to mention the yearbooks of The National Council of Teachers of Mathematics.

The reviewer feels, moreover, that many of the perplexities that beset the student of algebra originate in the pernicious persistence of the tradition to teach the method of one unknown for the solution of problems that really involve two or more unknowns. If students were taught from the very beginning of their study of algebra to solve verbal problems by the "multiple-equation" method many of their difficulties would disappear. This department is not, however, the place for the exposition of one's pet theories.

4. Georges, J. S. and Gorsline, W. W. *Nomography.* School Science and Mathematics. 36: 267-72. March 1936.

After pointing out that the method of intersecting coordinates is, in practice, limited to the case of two variables, the authors state that "if however we take for our coordinate axes a set of parallel lines, instead of intersecting lines, the inherent difficulty is eliminated, and the graphic method becomes of tremendous importance in the manipulation of complex formulas. The method of parallel axes was invented by D'Ocagne in 1884 to which he applied the name 'Nomography.' Since then a few books have been written on the subject, but the method has not been generally known or taught. It is felt that once the method is understood and its importance appreciated, it will find a proper place in the courses of mathematics in the high school and the college."

With the help of six simple examples and diagrams, the meaning and the value of nomography are clearly brought out.

Unfortunately, no bibliography is included.

5. Grossman, Howard D. *On the teaching of algebra.* High Points. May 1936, p. 74. Vol. 18, no. 5.

An interesting illustration of the possible correlation between mathematics and physics. The writer points out how a chandelier in the classroom, set swinging by a gust of wind, suggested an immediate application of square root. He also shows how topics in algebra may be interrelated more closely and more meaningfully.

6. Lippe, Adolph A. *Rules for the characteristic of a logarithm.* High Points. January 1936, pp. 61-63. Vol. 18, no. 1.

The writer claims that "the pupil who begins to use a logarithm table is frequently confused by a multiplicity of rules used to determine the characteristic of the logarithm of a number." He, therefore, proposes the following *one* rule to replace the customary rules.

- a. "Write the number whose logarithm you wish to find.
- b. "Consider units' place the 'zero' place for characteristic.
- c. "Count from units' place to the first significant figure of a number. (Significant figure means, in this discussion, any figure which is not zero.) The figure which is the result of your 'count' is the characteristic, positive if you have counted to the left, negative if you have counted to the right."

The corresponding rule for finding the anti-logarithm is also given, and the advantages of both rules over the traditional ones are indicated.

7. Woods, Roscoe. *The trigonometric functions of half or double an angle*. The American Mathematical Monthly. 43: 174-75. March 1936.

The author presents a simple geometric method for deriving the formulas connecting the trigonometric functions of double or half an angle, which "is not featured in texts on trigonometry."

#### Arithmetic

1. Dickey, John W. *Why invert the divisor and multiply?* School Science and Mathematics. 36: 299-302. March 1936.

The author criticizes the traditional explanation of the operation of division, and concludes that "the answer to the *why* of the inversion of the divisor and of the change of the operational sign in the division of fractions, is, that it renders the denominators common and non-effective, and places the numerators in an indicated division."

2. Grummann, H. R. *Shortened multiplication and division*. School Science and Mathematics. 36: 367-75. April 1936.

The writer presents the advantages to be derived from the algorithm for multiplication and division that was originated and developed by Dr. L. B. Tuckerman, now connected with the U. S. Bureau of Standards. Since it is impossible to summarize adequately the contents of an article of that nature, the reader is, therefore, referred to the original source.

#### Geometry

1. Carnahan, Walter H. *Geometrical constructions without the compasses*. School Science

and Mathematics. 36: 182-89. February 1936.

The following interesting statements are made:

- a. "If the compasses are discarded and every circle is lost and marking of the straight edge is prohibited, no geometric construction is possible.
- b. "If the compasses are discarded and we are permitted use of the marked straight edge, it is possible to make constructions involving determination of direction and marking off given lengths on the direction giving lines.
- c. "If the compasses are discarded but one fixed circle is given, it is possible with the straight edge alone to make every kind of geometric construction now made with both tools."

The author proceeds to illustrate statements b and c by giving detailed instructions and diagrams for eight constructions.

2. Lob, H. *Some geometrical applications of vectors*. The Mathematical Gazette. 20: 37-43. February 1936.

The writer illustrates "the straightforward way in which vectors lend themselves to geometry" by showing how they can be applied to the solution of the following problems:

- a. Relation between mutual distances of 4 points (or 5 points) in space.
- b. Circumsphere of a tetrahedron.
- c. Volume of a tetrahedron.
- d. Equation of sphere circumscribing the tetrahedron.
- e. Vector products in connection with tetrahedron, and the shortest distance between opposite edges of a tetrahedron.
- f. The sphere through the centroids of the faces.

#### Miscellaneous

1. Boyd, Rutherford. (a) *A mathematical theme in design*. (b) *Mathematical themes in ornament*. Scripta Mathematica. 4: 25, 36, 50. January 1936.

The first item listed is a full page photograph of a beautiful design in which all the lines are conic section or logarithmic spirals. The second item consists of five themes. The first three themes contain the following note: "Each theme is composed of arcs of one and the same logarithmic spiral. Note the rhythm resulting from geometric forms, of variant size and invariant shape."

2. Ginsburg, Jekuthiel. *Curiosa*. Scripta Mathematica. 4: 24. January 1936.

The new department, Curiosa, which the editor introduces in the January issue of *Scripta*

*Mathematica* should be of particular interest to those teachers of mathematics who are advisers of Mathematics Clubs or who like to relieve the serious moments of class room teaching with items of mathematical humor and history.

The Curiosa included in this issue are

- a. A problem in probability.
- b. Marat as a physicist.
- c. The term sine.
- d. 100 ways of writing 100.
- e. A quotation from Huxley.

A footnote invites readers "to submit items suitable for insertion under Curiosa."

3. Grimes, Ruby M. *Why mathematics?* School Science and Mathematics. 36: 426-37. April 1936.

A detailed enumeration of the various fields of knowledge in which the results of mathematics have been utilized. This article is one of the best on this subject that this reviewer has read, and should be made compulsory reading for the advocates of mathematics as well as for its opponents.

4. Grossman, Howard D. *Against math teams.* High Points. April 1936, p. 73. Vol. 18, no. 4.

A vigorous and justified attack on the competitive spirit in mathematics engendered by the New York Interscholastic Algebra League.

"It can not be denied that there is some good in mathematics teams. There is some genuine love and mastery of advanced mathematics, some genuine team-play and cooperation. But with the present stress on winning, the harm outweighs the good, and any step toward diminishing this stress is to the best interests of the students."

5. Heath, Royal V. *Concentric magic squares.* Scripta Mathematica. 4: 66-67. January 1936.

Full-page diagrams containing two interesting magic squares, one  $13 \times 13$ , the other  $11 \times 11$ , each one containing within it other magic squares.

6. Moritz, Robert E. *On the beauty of geometrical forms.* Scripta Mathematica. 4: 25-35. January 1936.

A clear and authoritative study of a fascinating subject. After a brief critical summary of various theories of beauty, the author concludes that "any acceptable theory of aesthetic measure applicable to geometric curves must be based on an analysis of their equations. The problem offers a challenge that may well en-

gage the attention of the aesthetician versed in the methods of mathematics, and the mathematician versed in the principles of aesthetics."

Two full-page plates, containing 24 curves and their equations, are included.

7. Moulton, E. J. *The future of mathematics.* School Science and Mathematics. 36: 124-37. February 1936.

The author summarizes his outlook for the future of mathematics in the following words:

- a. "Mathematical research will become more and more important to the world and will retain the respect of the community.
- b. "The school curriculum will be modified, but mathematics will retain an important place in it.
- c. "The teachers of the future will be more uniformly well prepared, their duties will be lightened, and there will be better mathematical instruction, insuring the perpetuity of a vigorous mathematical activity."

8. Pearson, Karl. *Old tripos days at Cambridge, as seen from another view point.* The Mathematical Gazette. 20: 27-36. February 1936.

The author feels that the article on the same subject that appeared in an earlier issue of *The Gazette* by Professor Forsyth painted the student life at Cambridge in the seventies in far too drab a color, and proceeds to give his own impression of them.

9. Richardson, R. G. D. *The Ph.D. degree and mathematical research.* The American Mathematical Monthly. 43: 199-215. April 1936.

A statistical study of the following topics:

- a. The number of Ph.D.'s conferred in the United States and Canada.
- b. Foreign influence on American Mathematics.
- c. Number of college teachers of mathematics.
- d. Proportion of Ph.D.'s publishing research papers.
- e. Comparison of productivity as regards universities of origin.
- f. Number of persons who have published papers but who have not attained to the doctor's degree.
- g. The publication rates for National Research Fellows.

Six statistical tables and two graphs are included.

10. Schenk, C. L. *A discussion of Sears, Roebuck and Company's thirteen period calendar*. School Science and Mathematics, 36: 163-73. February 1936.

An interesting analysis of the advantages to be derived from the universal acceptance of a thirteen month calendar. The causes and prejudices militating against such acceptance as well as the possible disadvantages are also pointed out.

11. Siddons, A. W. *Progress*. The Mathematical Gazette. 20: 7-26. February 1936.

The presidential address to the Mathematical Association (London, England), January 1936. It is an interesting historical study of the introduction of mathematical teaching into the English "public" schools, during the last century. This account should be of great interest to the historian, for it is replete with dates and stories of interesting personalities and contains extracts from contemporary documents.

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# NEWS NOTES

## Report on the Portland Meeting

The second summer meeting of the National Council of Teachers of Mathematics was held at Portland, Oregon, June 27, 28, and 29. The meetings were held at Reed College where dormitory and dining room accommodations were also available. The local committee had difficulty in getting publicity and a hotel for the convention because the Council is not affiliated with the N.E.A. They were delighted when Dr. F. L. Griffin invited them to meet at Reed College. This college is situated in a beautiful section of Portland. The campus is extensive, with green lawns, holly bushes and hedges, gigantic Douglas firs, and many other natural beauties. The buildings are of Tudor style, inspired by the architecture of Oxford University. The meals served in the College Commons were delicious. A more perfect place for a convention could not be found.

The program follows:

*Saturday, June 27, 10:00 a.m.*  
*Room 314, Eliot Hall*

Presiding: Miss Martha Hildebrandt

Attendance 95. Subject: Work of grades 7, 8, and 9.

1. Welcome to Oregon. Lesta Hoel, Portland, Oregon.
2. Mathematics: A Means of Expression (A Unit for Grade 8). Edith Woolsey, Minneapolis, Minn.
3. Meeting the Educational Needs of Non-college Preparatory Students in Washington (Grade 9). Paul Wright, Davenport, Washington.
4. Not "One-sided" but "All-round" Improvement in Junior High School Mathematics. Edith L. Mossman, Berkeley, Calif.
5. Round Table discussion of class-room problems.

*Saturday, June 27, 2:00 p.m.*  
*Room 221, Eliot Hall*

Arithmetic Section

Presiding: Dr. F. L. Griffin

Attendance 28

1. The Meaningful Teaching of Mechanical Phases of Arithmetic. Dr. H. C. Christoferson, Oxford, Ohio.
2. Recent Changes of Point of View Relating to the Teaching of Arithmetic. Dr. E. A. Bond, Bellingham, Washington.

3. Round Table discussion led by E. L. McDonnell, Seattle, Washington. Mrs. Jane Goddard, Portland, Oregon

*Room 314, Eliot Hall*  
*Work of Grades 10, 11, and 12*

Presiding: Miss Martha Hildebrandt.

Attendance 91.

1. Selling Mathematics, Anna M. Whitney, Yakima, Washington.
2. Marking Elementary Algebra Function in Advance High School Courses. Elsie Parker Johnson, Oak Park, Illinois.
3. Round Table discussion: How I make mathematics an alive and useful subject in and out of the classroom.

Discussion led by:

Sarah Ruby, Portland, Oregon  
Beryl Holt, Salem Oregon  
Alfred Davis, St. Louis, Missouri  
Kate Bell, Spokane, Washington  
Alice M. Reeve, Rockville Center, New York  
Annabel Donneley, Okmulgee, Oklahoma  
Nellie B. Wilkinson, Phoenix, Arizona

## Banquet

*Saturday, June 27, 6:30 p.m.*  
*Reed College Commons*

Attendance 103.

The guests were seated at small tables. A gorgeous centerpiece of flowers decorated each table and a rose marked each place. The folders for the banquet programs were furnished by the American Mail Line, Orient via Seattle. The programs were designed in Oriental style by Alfred B. Carter of Grant High School.

Dinner music: Grant High School students through the courtesy of the Mathematics Department of Grant High School: John Van Gorder, Violin solo; Harry Steele, vocal solo; Margaret Mullin, accompanist.

1. Miss Martha Hildebrandt asked Miss Lesta Hoel, chairman of the local committee, to introduce the other members of the committee.
2. Miss Hildebrandt then called on Miss Edith Woolsey, secretary pro tem., for a report on the registration. She announced that 128 had registered from 16 states. Representatives from each state were called on to stand.
3. Welcome to Reed College. President Dexter M. Keezer of Reed College: Dr. F. L.

Griffin, head of the Mathematics Department of Reed College.

4. Address: A Mathematical Genius and His Teachers. Dr. Eric Temple Bell, California Institute of Technology, Pasadena, California

*Sunday, June 28, 10:00 a.m.*

*Outing to Bonneville Dam*

Transportation was furnished by the Portland mathematics teachers for all who were registered. The trip took us out the Columbia River Highway on the Portland side and back on the Washington side. Stops were made at various points of interest, the best known of which was Multnomah Falls. This trip was through that gorgeous, unsurpassed natural scenery of which the residents of Oregon are so justly proud. The visit to the Bonneville Dam was tremendously interesting. The caravan returned to Portland in time for the Vesper Service of the N.E.A. convention.

*Monday, June 29, 9:00 a.m.*

*Room 314, Eliot Hall*

General Meeting.

Presiding: Miss Martha Hildebrandt

Attendance 60.

1. Placing College Freshmen in Mathematics, Dr. W. E. Milne, Oregon State College, Corvallis, Oregon.
2. Mathematics: A Tool Subject or a System of Thought. Dr. F. L. Griffin, Portland, Oregon.
3. Discussion

*Monday, June 29, 3:00 p.m. Chapel of First Baptist Church.*

Joint Conference: Department of Secondary Education in cooperation with the National Council of Teachers of Mathematics.

Attendance 170.

Miss Agnes Beach presided but called upon Miss Martha Hildebrandt to introduce the speakers.

Central Topic: Trends in Mathematics Instruction in High School and Possible Use of New Curriculum Materials.

1. Dr. H. C. Christofferson, Miami University, Oxford, Ohio.
2. Mr. Earl Murray, Santa Barbara, California.
3. Discussion led by Prof. E. E. DeCou, University of Oregon, Eugene, Oregon.

Following this meeting, everyone attending was invited to meet outside at a stated place, where a picture was taken. Copies of this picture may be bought for one dollar. Communicate with Dr. F. L. Griffin, Reed College.

Local Committee:

Lesta Hoel, chairman

Dr. F. L. Griffin, head of the Department of Mathematics, Reed College

A. F. Bittner, Principal, Grant High School

Abigail McRaith, Rena Anderson, H. J.

Nottage, Cora Shaver, Lee A. Dillon, Leo

Oldwright, Henry Baldwin.

#### ATTENDANCE

|                   |                          |
|-------------------|--------------------------|
| <i>Alaska</i>     |                          |
| Juneau            | Marjorie Tillotson (W)   |
| <i>Arizona</i>    |                          |
| Phoenix           | Nellie B. Wilkinson (G)  |
| <i>California</i> |                          |
| Berkeley          | Edith Mossman (W)        |
| Boulder Creek     | Mrs. Stella Hesse (G)    |
| Escondido         | Hazel Bolton (W)         |
| Oakland           | Emma Hesse (W)           |
|                   | Helen Munny (W)          |
| Pasadena          | E. T. Bell (G)           |
|                   | Mrs. E. T. Bell (G)      |
| Roseville         | Elsie Dunn (W)           |
| Sacramento        | Frances Charters (G)     |
| Santa Barbara     | Earl Murray (W)          |
|                   | Mrs. Earl Murray (G)     |
| Whittier          | Mattie A. Gregg (G)      |
| <i>Colorado</i>   |                          |
| Denver            | L. Denzil Keigley (R)    |
| <i>Hawaii</i>     |                          |
| Honolulu          | Lilla Annis (W)          |
| <i>Illinois</i>   |                          |
| Chicago           | Mabelle Dhus (W)         |
| Maywood           | Martha Hildebrandt (B)   |
| Normal            | Edith I. Atkin (R)       |
| Oak Park          | Elsie Parker Johnson (B) |
| Wauconda          | Beulah Steele (G)        |
| Waukegan          | Bess L. Dady (R)         |
|                   | Margaret Dady (R)        |
| Witt              | Cora Lipe (W)            |
| Yorkville         | Ester Roesch (W)         |
| <i>Iowa</i>       |                          |
| Anamosa           | Lilly M. Rodine (W)      |
| <i>Maryland</i>   |                          |
| Annapolis         | Randolph Church (G)      |
| <i>Minnesota</i>  |                          |
| Mankato           | Alice Robbins (R)        |
| Minneapolis       | Blanche Berg (G)         |
|                   | Lucile Moorman (G)       |
|                   | Edith Woolsey (R)        |
| <i>Missouri</i>   |                          |
| St. Louis         | Alfred Davis (R)         |
| <i>Nebraska</i>   |                          |
| Bridgeport        | Helen Boentje (G)        |
| <i>New York</i>   |                          |
| Rochester         | Paul J. Smith (W)        |
| Rockville Center  | Alice M. Reeve (R)       |
| <i>Ohio</i>       |                          |
| Marion            | Raymond McNutt (W)       |
| Oxford            | H. C. Christofferson (R) |
| <i>Oklahoma</i>   |                          |
| Okmulgee          | Mattie Bogue (R)         |
|                   | Annabel Donneley (R)     |
| <i>Oregon</i>     |                          |
| Astoria           | Minnie Ambler (W)        |
|                   | Esther Larson (W)        |
| Baker             | Ella Moulton (G)         |

